Review Definition of matrix multiplication

A $m \times n$ matrix
B $n \times p$ matrix $= [\boldsymbol{b_1} \quad \boldsymbol{b_2} \quad \boldsymbol{b_p}]$

Define $AB = [A\boldsymbol{b_1}\ A\boldsymbol{b_2} \ldots A\,\boldsymbol{b_p}]$

$\uparrow$
$m \times p$

Example: $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\boldsymbol{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\boldsymbol{u}^T = [1\ 2\ 3]$, $\boldsymbol{v}^T = [4\ 5\ 6]$

$$\boldsymbol{v}^T\boldsymbol{u} = [4\ 5\ 6]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [32]$$ (a 1×1 matrix is just written as a number, 32)

$\uparrow$ inner product (or "dot product") of vectors $\boldsymbol{u}, \boldsymbol{v}$ (written as a matrix product(

$$\boldsymbol{u}\boldsymbol{v}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}[4\ 5\ 6] = \left[\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot 4 \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot 5 \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot 6\right] = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

$\uparrow$ outer product of vectors $\boldsymbol{u}$, $\boldsymbol{v}$

Example:

$$\underset{A}{\begin{bmatrix} 0 & 2 & 1 \\ 3 & 4 & 1 \\ \mathbf{2} & \mathbf{1} & \mathbf{4} \end{bmatrix}} \underset{B=[\boldsymbol{b_1}\ \boldsymbol{b_2}\ \boldsymbol{b_3}\ \boldsymbol{b_4}]}{\begin{bmatrix} 1 & \mathbf{2} & 0 & 1 \\ 1 & \mathbf{3} & 1 & 0 \\ 0 & \mathbf{1} & 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} A\begin{bmatrix}1\\1\\0\end{bmatrix} & A\begin{bmatrix}2\\3\\1\end{bmatrix} & A\begin{bmatrix}0\\1\\0\end{bmatrix} & A\begin{bmatrix}1\\0\\1\end{bmatrix} \end{bmatrix}$$

$$= \ \ldots \ \begin{bmatrix} 2 & 7 & 2 & 1 \\ 7 & 19 & 4 & 4 \\ 3 & \mathbf{11} & 1 & 6 \end{bmatrix} = C$$

$$\uparrow$$

$$\mathbf{2(2)+1(3)+4(1)}$$

$$\boldsymbol{c_{3,2}} = (\mathbf{row}_3 A)(\mathbf{col}_2 B) \quad \text{("row-column rule" for entries in } C\text{)} \quad \text{and}$$

$$= (\mathbf{row}_3 A)\, \boldsymbol{b_2}$$

$$(\mathbf{row}_3 A)B = (\mathbf{row}_3 A)[\, \boldsymbol{b_1} \ldots \boldsymbol{b_4}]$$

$$= [(\mathbf{row}_3 A)\boldsymbol{b_1} \ \ldots \ (\mathbf{row}_3 A)\boldsymbol{b_4}]$$

$$= [\ \boldsymbol{c_{31}}\ \boldsymbol{c_{32}}\ \boldsymbol{c_{33}}\ \boldsymbol{c_{34}}] = \mathbf{row}_3(AB)$$

Example (same A, B)

$$[\mathbf{2\,1\,4}]\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} = [\mathbf{3}\ \ \mathbf{11}\ \ \mathbf{1}\ \ \mathbf{6}]$$

In general,

$$\boldsymbol{c_{i,j}} = (\mathbf{row}_i A)(\mathbf{col}_j B)$$

$$\mathbf{row}_i(AB) = (\mathbf{row}_i A)B$$

Properties of Multiplication of Matrices

If A, B, C are all the right sizes for each equation:

1. $A(BC) = (AB)C$ because:

   Suppose A is $m \times n$, B is $n \times p$
   C is $p \times q$, say $C = [\boldsymbol{c_1} \ldots \boldsymbol{c_q}]$

   Then $A(BC) = A[B\boldsymbol{c_1} \quad B\boldsymbol{c_2} \quad \ldots B\boldsymbol{c_q}]$
   $= [AB\boldsymbol{c_1} \quad AB\boldsymbol{c_2} \quad \ldots AB\boldsymbol{c_q}]$
   $\nearrow$
   $\searrow$
   and $(AB)C = [AB\boldsymbol{c_1} \quad AB\boldsymbol{c_2} \quad \ldots \quad AB\boldsymbol{c_q}]$

2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$ (note the order: BA, not AB, etc.)
4. $r(AB) = (rA)B = A(rB)$

5. $I_m A = A = AI_n$
   $\nwarrow \qquad \nearrow$
   when A is $m \times n$

6. $(AB)^T = B^T A^T$ *(see next page)*

To verify 6:

If A is $m \times n$ and B is $n \times p$, then $(AB)^T = B^T A^T$

?? Why ??

AB is $m \times p$, so $\quad (AB)^T$ **is** $p \times m$

B^T is $p \times n$,

A^T is $n \times m$, so $\quad B^T A^T$ **is** $p \times m$

So $(AB)^T$ aSond $B^T A^T$ **are same size,** but are they $=$?

Compare their (i, j) entries to see if they're the same:

$$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \vdots & \vdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \leftarrow \mathbf{Row}_j A = \mathbf{Col}_j A^T$$

$$\mathbf{Col}_i B = \mathbf{Row}_i B^T$$

$$B = \begin{bmatrix} \vdots & \vdots & \vdots & \downarrow & \vdots & \vdots \\ \vdots & \vdots & \vdots & b_{1i} & \vdots & \vdots \\ \vdots & \vdots & \vdots & b_{2i} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & b_{ni} & \vdots & \vdots \end{bmatrix}$$

(i, j) entry in $(AB)^T$

$= (j, i)$ entry in B

$= \mathbf{Row}_j A \cdot \mathbf{Col}_i B \qquad = a_{j1}b_{1i} + a_{j2}b_{2i} + \ldots + a_{jn}b_{ni}$

(i, j) entry in $B^T A^T$

$= \mathbf{Row}_i B^T \cdot \mathbf{Col}_j A^T \qquad = b_{1i}a_{j1} + b_{2i}a_{j2} + .. + b_{ni}a_{jn}$

$= a_{j1}b_{1i} + a_{j2}b_{2i} + \ldots + a_{jn}b_{ni}$

So **(i, j) entry in $(AB)^T$** $=$ **(i, j) entry in $B^T A^T$**

Definition of inverse for a square matrix A and some properties of inverses: discussed in class

Suppose A, B are both $n \times n$"

1) If A, B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
2) If A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$
3) If A is ivertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$

"Magic" Rule to determine if $n \times n$ matrix is invertible and, if it is, to find A^{-1}
(we will see why it works and remove the "magic" in next lecture)

Put A and I_n side by side

$$[\, A \mid I_n \,]$$

On the whole matrix, perform those EROs needed to reduce A to rref (the resulting changes in I_n are just carried along throughout)

Finally, we get to

$$\begin{array}{cc} A & I_n \\ \sim \downarrow & \sim \downarrow \\ [\ \text{rref } A & \text{some matrix } C\ ] \end{array}$$

- **If rref $A = I_n$, then A is invertible and the matrix C is A^{-1}**
- **If rref $A \neq I_n$, then A is not invertible (and C is irrelevant)**