Test yourself (*not mentioned in class*)

1. a) If the variables are $y, v, w,$ and $t$ (and $x, u, \pi,$ and $z$ are constants) then the equation is linear

$$xy + 6uv - \pi zw = 8t - 22$$

b) If the variables are just $x$ and $z$, and all the other symbols are constants, then the equation is linear.

$$xy + 6uv - \pi zw = 8t - 22$$

c) If $x$ and $y$ are variables, then the equation is not linear.

$$xy + 6uv - \pi zw = 8t - 22$$

2. Solving a system

$$ax_1 + bx_2 = 5$$
$$cx_1 + dx_2 = 6$$

Suppose we get that

$$x_1 = 3$$
$$x_2 = 5$$

Is that one solution, or two?

3. $$\begin{bmatrix} 1 & 7 & 0 & 3 & 5 \\ 1 & 0 & 1 & 0 & 16 \\ 2 & 14 & 0 & 0 & 10 \end{bmatrix} \sim \; ... \; \sim \begin{bmatrix} 1 & 7 & 0 & 3 & 5 \\ 2 & 0 & 2 & 0 & 32 \\ 0 & 0 & 0 & -6 & 0 \end{bmatrix}$$

Which row operation was not done?

a) (row 2) $\rightarrow$ 2(row 2)

b) interchange (row 2), (row 3)

c) (row 3) $\rightarrow$ (row 3) $-$ 2(row 1)

**A matrix is in an echelon form if:**

**1) all rows that contain only 0's are moved to be the bottom rows of the matrix, and**

**2) in the nonzero rows: the leading entries ( = first nonzero entries) are always in columns further to the right as you go down the rows, and**

**3) all entries in a column below a leading entry are 0's**

**The matrix is in its reduced row echelon form ( "rref" ) if also**

**4) in each nonzero row, the leading entry = 1,**

**5) each leading 1 has only 0's above and below it.**

**An echelon form** and **its corresponding rref**

**where**

■ = a nonzero number
$*$ = any number (including perhaps 0)

$$\begin{bmatrix} \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

WHICH OF THE FOLLOWING ARE IN REDUCED ROW ECHELON FORM (RREF)?

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 17 & 0 & 13 & 5 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## The Row Reduction Algorithm (see application in example that follows)

### Part I: Forward phase

1) Locate leftmost nonzero column

2) Interchange rows (if necessary) to get a nonzero element in the top row of that column (a pivot position)

3) Use EROs to create 0's in all positions below that pivot position

4) Ignore the row containing the pivot position and all rows above it.

Apply 1-3 again to the remaining submatrix. Continue this "loop Steps 1-3" until you have an echelon matrix.

**Part II: Backward phase** Apply Step 5) for each pivot, starting at the lowest rightmost pivot, working backward and up toward the left:

5) Use row rescaling, if necessary, to create a 1 in the pivot position; then use EROs to create 0's above this pivot (OR, if it makes the arithmetic easier, first use row replacements to create 0's above each pivot position, starting at the lower right and working backward and up toward the left, and then, when all that is finished, rescale the leading entries to 1's.

(*In general for large matrices, the "backward" phase involves considerably less arithmetic than the "forward" phase – see comments in textbook*)

**<u>Row Reduction Example</u> (watch how each step follows the row reducation steps above)**
**Descibre the ERO used at each step.**

$$\begin{aligned} x_1 + x_2 \; + \; x_3 &= \;\; 1 \\ 2x_1 + 3x_2 - \; x_3 &= -4 \\ -x_1 + 2x_2 + \; x_3 &= \;\; 3 \end{aligned}$$

$$\text{Augmented matrix} = \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 \\ 2 & \mathbf{3} & -1 & -4 \\ -1 & 2 & \mathbf{1} & 3 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{1} & -3 & -6 \\ -1 & 2 & \mathbf{1} & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{1} & -3 & -6 \\ 0 & 3 & \mathbf{2} & 4 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{1} & -3 & -6 \\ 0 & 0 & \mathbf{11} & 22 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{1} & -3 & -6 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix}$$

↑

*This is in echelon form, as is each matrix that follows: a nonzero matrix has many different echelon forms, but it has <u>one and only one rref</u> : a theorem we don't prove.*

$$\sim \begin{bmatrix} \mathbf{1} & 1 & 1 & 1 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 1 & 0 & -1 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 0 & 0 & -1 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix}$$

↑

(*this is <u>the rref</u>: the <u>unique</u> row reduced echelon form*

<u>Pivot positions</u> are positions that contain a leading 1 in rref form. The pivot positions become "visible" earlier in the echelon forms – positions where there are nonzero leading entries in Pivot positions above highlighted in red. Pivot positions not usually clear before finding an echelon form.

**<u>Pivot column</u>**: a column that contains a pivot position
**<u>Basic variable</u>**: a variable corresponding to a pivot column;
other variables (if any) are called **<u>free variables</u>**

<u>In rref</u>, a nonzero entry in a row and sitting to the <u>right</u> of a leading 1 must correspond to a free variable or (in the last column of an augmented matrix) a constant. (*Why?*). Therefore we can look at the rref and solve for each basic variable in terms of the free variables and constant to its right). In the example:

Pivot columns are columns $1, 2, 3$ and the basic variables are $x_1, x_2, x_3$ there are no free variable

$$\text{We can solve as} \begin{cases} x_1 = -1 \\ x_2 = 0 \\ x = 2 \end{cases}$$

**Example** Suppose we have an augmented $A$ for which

$$A = ... \sim \text{rref}(A) = \begin{bmatrix} \mathbf{1} & 1 & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2 pivot columns corresponding to basis variables $x_1$ and $x_3$; Column 2 is not a pivot column so $x_2$ is a free variable.

Solve for basic variables in terms of free variable $x_2$ :

$x_1 = -1 - x_2$
$x_2$ free (*could also express this as* $x_2 = x_2$*, meaning "*$x_2$ *is what it is"* ).
$x_3 = 2$

This form of solution is sometimes called <u>parametric</u> since the answer is in terms of a "parameter" $x_2$.

**Example** Suppose we have an augmented $A$ for which

$$A = ... \sim \text{rref}(A) = \begin{bmatrix} \mathbf{1} & -3 & 4 & -4 \\ 0 & \mathbf{1} & -\frac{5}{2} & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$x_1$ and $x_2$ are basic variables; $x_3$ is free.

The bottom row shows that this system is inconsistent: $0x_1 + 0x_2 + 0x_3 = 3$ is impossible.

<u>This illustrates that an inconsistent system can have a free variable</u>.

A system is inconsistent if and only if a row like [0,0, ... ■), where ■ $\neq 0$, appears in an echelon form of $A$ (that is, if an only if the rightmost column of the augmented matrix is a pivot column)

**Example** Suppose we have an augmented $A$ for which

$$A = [1 \ 2 \ 3 \ 4]$$

The augmented matrix is already in rref; it corresponds to the system

$x_2 + 2x_2 + 3x_3 = 4$ (system with one equation, 4 unknowns)

$x_1$ is a basic variable and $x_2, x_3$ are free. Solve for the basic variable in terms of the free variables:

$$x_1 = 4 - 2x_2 - 3x_3$$
$$x_2, x_3 \text{ are free}$$

Sometimes a writer will "rename" the parameters and write something like

$$x_1 = 4 - 2s - 3t$$
$$x_2 = s$$
$$x_3 = t \quad \text{where } s, t \text{ are free}$$

This is another way to say the same thing,

---

**Theorem** For any linear system of equations, either

i) the system is inconsistent (if and only if a row like [0,0, ... ■) appears in an echelon form of the augmented matrix)

or ii) the system is consistent with

a) exactly one solution if the system has <u>no</u> free variables, or
b) infinitely many solutions if the system has one or more free variables

(So, for example, a system of linear equations <u>cannot</u> have exactly two solutions.)

**EXAMPLE Chemical Reaction (see Section 1.6)**

boron sulfide + water ----> boric acid + hydrogen sulfide

$(x_1)$ $\mathbf{B_2S_3}$ + $(x_2)$ $\mathbf{H_2O}$----> $(x_3)$$\mathbf{H_3BO_3}$ + $(x_4)$ $\mathbf{H_2S}$

Same number of <u>boron</u> atoms on both sides so:

$$2x_1 \qquad = x_3$$

Same number of <u>sulfur</u> atoms on both sides:

$$3x_1 \qquad = x_4$$

Same number of <u>hydrogen</u> atoms on both sides:

$$2x_2 \qquad = 3x_3 + 2x_4$$

Same number of <u>oxygen</u> atoms on both sides:

$$x_2 \qquad = 3x_3$$

So to "balance", we need to satisfy all the equations:

$$\begin{aligned} 2x_1 \qquad - x_3 \qquad\qquad &= 0 \\ 3x_1 \qquad\qquad\qquad x_4 \; &= 0 \\ 2x_2 - 3x_3 - 2x_4 \; &= 0 \\ x_2 - 3x_3 \qquad\qquad &= 0 \end{aligned}$$

Row Reduction Steps (Identify the ERO at each step)

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 0 & 2 & -3 & -2 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 0 & 2 & -3 & -2 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 3/2 & -1 & 0 \\ 0 & 2 & -3 & -2 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 2 & -3 & -2 & 0 \\ 0 & 0 & 3/2 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 3/2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$. Basic variables $x_1, x_2.\ x_3;\ x_4$ free

$$\begin{cases} x_1 \qquad\qquad - \frac{1}{3}\, x_4 = 0 \\ \quad x_2 \qquad - 2\ x_4 = 0 \\ \qquad x_3 - \frac{2}{3} x_4 = 0 \\ \qquad\qquad 0 = 0 \end{cases}$$

$x_1 = \frac{1}{3} x_4,$
$x_2 = 2x_4,$
$x_3 = \frac{2}{3} x_4$
$x_4$ is free (meaning $x_4$ can have <u>any</u> value: we could write that as $x_4 = x_4$ meaning "$x_4$ is what it is")

So: mathematically, there are infinitely many solutions. But <u>chemically</u> we want the $x_i$'s to be positive integers, so we choose $x_4$ to be a multiple of 3.

If we $x_4 = 3$, then

$x_1 = 1,\ x_2 = 6,\ x_3 = 2,$ so

$B_2S_3$ + 6 $H_2O$ ----> 2 $H_3BO_3$ + 3 $H_2S$ is balanced.

(If we chose $x_4 = 6$, we'd get

2 $B_2S_3$ + 12 $H_2O$ ----> 4 $H_3BO_3$ + 6 $H_2S$ – also balanced and physically correct, but the chemist would probably want to use the smallest set of coefficients work.