VERSION 1, with 2 PARTS (previously done): suppose $\boldsymbol{T}$ is a linear function, where $\boldsymbol{T} : \mathbb{R}^{\boldsymbol{n}} \longrightarrow \mathbb{R}^{\boldsymbol{m}}$, with standard matrix $\boldsymbol{A}$

T is onto: that is, for every $\boldsymbol{b} \in \mathbb{R}^m$ there is at least one $\boldsymbol{x} \in \mathbb{R}^n$ for which $T(\boldsymbol{x}) = \boldsymbol{b}$	$\Leftrightarrow$	For every $\boldsymbol{b} \in \mathbb{R}^m$, there is at least one $\boldsymbol{x} \in \mathbb{R}^n$ for which $A\boldsymbol{x} = \boldsymbol{b}$ (that is, the mapping $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is onto
$\Updownarrow$		$\Updownarrow$ (*see Theorem 4, p. 43*)
For every $\boldsymbol{b} \in \mathbb{R}^m$, the equation $T(\boldsymbol{x}) = \boldsymbol{b}$ has at least one solution		Every vector $\boldsymbol{b} \in \mathbb{R}^m$ is a linear combination of the columns of A
		$\Updownarrow$ (*See Theorem 4, p. 43*)
		The columns of A span $\mathbb{R}^m$
		$\Updownarrow$ (*see Theorem 4, p. 43*)
		A has a pivot in every row.

T is one-to-one: that is, for every $\boldsymbol{b} \in \mathbb{R}^m$ there is at most one $\boldsymbol{x} \in \mathbb{R}^n$ for which $T(\boldsymbol{x}) = \boldsymbol{b}$.	$\Leftrightarrow$	For every $\boldsymbol{b} \in \mathbb{R}^m$, there is at most one $\boldsymbol{x} \in \mathbb{R}^n$ for which $A\boldsymbol{x} = \boldsymbol{b}$.
$\Updownarrow$		$\Updownarrow$
For every $\boldsymbol{b} \in \mathbb{R}^m$, the equation $T(\boldsymbol{x}) = \boldsymbol{b}$ has at most one solution.		For every $\boldsymbol{b} \in \mathbb{R}^m$, the equation $\boldsymbol{A}x = \boldsymbol{b}$ has at most one solution.
$\Updownarrow$ (*See Theorem 11, p. 88*)		$\Updownarrow$
The equation $T(\boldsymbol{x}) = \mathbf{0}$ has only the trivial solution (namely, $\boldsymbol{x} = \mathbf{0}$)	$\Leftrightarrow$	The homogeneous system $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution $\boldsymbol{x} = \mathbf{0}$.
		$\Updownarrow$
		The system $A\boldsymbol{x} = \mathbf{0}$ has no free variables (that is, every column of A is a pivot column)
		$\Updownarrow$ (*See Statement (3), p. 66*)
		The columns of A are linearly independent

VERSION 2: We showed earlier that all the statements within each box (see other side) are equivalent. Now we ask **what more is true if we also assume that $m = n$.** In that case, we have that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping with **square** standard matrix $A_{n\times n}$. Because A is square, it then turns that out all the statements on this page are equivalent. including some additional statements (not on the other side) which appear **in boldface** are equivalent. Version 2 – the statement that these are all equivalent – is called the **Invertible Matrix Theorem (IMT)** in the textbook.

T is onto: that is, for every $\boldsymbol{b} \in \mathbb{R}^n$ there is at least one $\boldsymbol{x} \in \mathbb{R}^n$ for which $T(\boldsymbol{x}) = \boldsymbol{b}$ For every $\boldsymbol{b} \in \mathbb{R}^n$, the equation $T(\boldsymbol{x}) = \boldsymbol{b}$ has at least one solution	For every $\boldsymbol{b} \in \mathbb{R}^n$, there is at least one $\boldsymbol{x} \in \mathbb{R}^n$ for which $A\boldsymbol{x} = \boldsymbol{b}$ (that is, the mapping $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is onto) Every vector $\boldsymbol{b} \in \mathbb{R}^n$ is a linear combination of the columns of A The columns of A span $\mathbb{R}^n$ A has a pivot in every row

$\Updownarrow$ $\leftarrow$ **because A is $n \times n$**

A has a pivot in every column $\Leftrightarrow$ **A has exactly n pivots**

$\Updownarrow$

rref $A = I_n$

$\Updownarrow$

A^T is invertible $\Leftrightarrow$ **A is invertible**

$\Updownarrow$

The equation $T(\boldsymbol{x}) = \boldsymbol{0}$ has only the trivial solution (namely, $\boldsymbol{x} = \boldsymbol{0}$) For every $\boldsymbol{b} \in \mathbb{R}^n$, the equation $T(\boldsymbol{x}) = \boldsymbol{b}$ has at most one solution. T one-to-one, that is, for every $\boldsymbol{b} \in \mathbb{R}^n$ there is at most one $\boldsymbol{x} \in \mathbb{R}^n$ for which $T(\boldsymbol{x}) = \boldsymbol{b}$.	The homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$. For every $\boldsymbol{b} \in \mathbb{R}^n$, the equation $\boldsymbol{A}x = \boldsymbol{b}$ has at most one solution. For every $\boldsymbol{b} \in \mathbb{R}^n$, there is at most one $\boldsymbol{x} \in \mathbb{R}^n$ for which $A\boldsymbol{x} = \boldsymbol{b}$ The columns of A are linearly independent

$\Updownarrow$

T is an invertible transformation: that is there is a linear $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $(T \circ S)(x) = x = (S \circ T)(x)$ for every x in $\mathbb{R}^n$

$\Updownarrow$

There is an $n \times n$ matrix C such that $CA = I_n$

$\Updownarrow$

There is an $n \times n$ matrix D such that $AD = I_n$