Diagonalization Example

Can we diagonalize: $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$? That is, can we factor

$A = P_{\mathcal{B}} D P_{\mathcal{B}}^{-1}$ for some basis $\mathcal{B}$ and some diagonal matrix D ?

The answer is "yes" <u>if</u> we can find a basis consisting of eigenvectors of A.
(See the notes: *Introduction to Diagonalization* from the preceding lecture.)

We need see, first if we can find any eigenvectors at all. An eigenvector is a <u>nonzero</u> solution to $A\boldsymbol{x} = \lambda \boldsymbol{x}$ (where λ can be any scalar). So, if this is possible, there are really two "unknown" items at this point: the λ, and the corresponding $\boldsymbol{x}$'s.

We can rewrite $A\boldsymbol{x} = \lambda\boldsymbol{x}$ as $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ where I is the identity matrix.
This <u>homogeneous</u> equation has a <u>nonzero</u> (=nontrivial) solution for $\boldsymbol{x}$ if and only if $\det(A - \lambda I) = 0$. We use this fact first to find the possible λ's (eigenvalues); then, for each possible λ, separately, we find the corresponding $\boldsymbol{x}$'s (eigenvectors)

For the given matrix A:

(*) $\quad (A - \lambda I)\boldsymbol{x} = \mathbf{0}$

$$\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} \boldsymbol{x} = \mathbf{0}$$

$$\det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = -(2-\lambda)(6+\lambda) - 9$$

$$= \lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3) = 0 \text{ so } \lambda = -7 \text{ or } 7.$$

(*) will have <u>nonzero</u> solutions for $\boldsymbol{x}$ if and only if $\lambda = -7$ or $\lambda = 3$.

<u>For $\lambda = -7$</u> :

$$(A + 7I)\boldsymbol{x} = \mathbf{0}$$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix} \text{ so } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

By rescaling, we can rewrite the general solution (for neatness) as $\boldsymbol{x} = s\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

So, then eigenvectors corresponding to $\lambda = -7$ are the nonzero multiples of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

<u>For $\lambda = 3$</u>:

$(A - 3I)\boldsymbol{x} = \mathbf{0}$

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

So, then eigenvectors corresponding to $\lambda = 3$are the nonzero multiples of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

If we pick one eigenvector for each eigenvalue, we can get a basis (consisting of eigenvectors) for $\mathbb{R}^2$. $\mathcal{B} = \{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \}$

According to the notes, we can diagonalize A as

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}}_{P_{\mathcal{B}}} \underbrace{\begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{1}{10} & -\frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}}_{P_{\mathcal{B}}^{-1}}$$

where the diagonal matrix D is created using the eigenvalues for the eigenvectors in the columns of $P_{\mathcal{B}}$ (in the same order).

An additional observation about what diagonalization can be useful in computations:

If $\quad A = PDP^{-1}$, then

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) = PD^2P^{-1} \\ A^3 &= A^2A = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1} \\ &\vdots \\ A^n &= A^{n-1}A = (PD^{n-1}P^{-1})(PDP^{-1}) = PD^nP^{-1} \end{aligned}$$

This makes computing powers of A much easier: very handy if n is large, and even more so if A is a very large sized matrix. A "small" example, from above:

$$A^{20} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}^{20} \begin{bmatrix} \frac{1}{10} & -\frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix} = \; = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 7^{20} & 0 \\ 0 & 3^{20} \end{bmatrix} \begin{bmatrix} \frac{1}{10} & -\frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

1.0e+016 * $\longleftarrow$ **Matlab's notation: each entry in the displayed matrix is by 10^{16} (*of course, there's roundoff in Matlab's answer*)**

0.79792297678672	**-2.39376788432483**
-2.39376788432483	**7.18130400165292**