Remember: We observed that for <u>any</u> two matrices A, B with sizes so that AB is defined:

$$\text{row}_i(AB) = \text{row}_i(A) \cdot B \quad (\textit{we use this over \& over, below})$$

Theorem Multiplying a matrix A, on the left, by an elementary matrix E (*of the correct size*) performs <u>the same</u> elementary row operation on A as was used to create E from I.

> *To keep the notation simple, we assume A is a 3×3 matrix and that I is the 3×3 identity matrix.*
>
> *So the following is not exactly a proof (because it's restricted to just the 3×3 case. But <u>exactly</u> the same method are used in the case where A is $m \times n$ and I is the $m \times m$ identity matrix: it's just a bit harder to write it all down in general. If you understand the 3×3 situation, then you'll understand why it always works.*

"Proof" There are three kinds of ERO that might be used to create E from I_3. We look at each one, separately, in order to see why the theorem is true for that kind of elementary matrix E.

1) <u>Add a multiple of one row to another</u> Suppose, for example, that E is obtained from $I = I_3$ by adding -2(row 1) to row 3

$$E = \begin{bmatrix} \text{row}_1\, E \\ \text{row}_2\, E \\ \text{row}_3\, E \end{bmatrix} = \begin{bmatrix} \text{row}_1\, I \\ \text{row}_2\, I \\ -2(\text{row}_1\, I) + \text{row}_3\, I \end{bmatrix}$$

$$\text{Then } EA = \begin{bmatrix} \text{row}_1\, EA \\ \text{row}_2\, EA \\ \text{row}_3\, EA \end{bmatrix} = \begin{bmatrix} (\text{row}_1 E)A \\ (\text{row}_2 E)A \\ (\text{row}_3 E)A \end{bmatrix} = \begin{bmatrix} (\text{row}_1 I)A \\ (\text{row}_2 I)A \\ (-2(\text{row}_1\, I) + \text{row}_3\, I)A \end{bmatrix}$$

$$= \begin{bmatrix} \text{row}_1(IA) \\ \text{row}_2(IA) \\ -2(\text{row}_1\, I)A + \text{row}_3\,(I)A \end{bmatrix} = \begin{bmatrix} \text{row}_1(IA) \\ \text{row}_2(IA) \\ -2\,\text{row}_1(\,IA) + \text{row}_3\,(IA) \end{bmatrix}$$

$$= \begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ -2\,\text{row}_1(A) + \text{row}_3\,(A) \end{bmatrix}.$$ So the effect of computing EA is

the same as performing the row operation "add -2(row 1) to row 3" on A.

2) <u>Interchange two rows</u> Suppose, for example, that E is obtained from I by interchanging rows 1 and 3

$$E = \begin{bmatrix} \text{row}_1\, E \\ \text{row}_2\, E \\ \text{row}_3\, E \end{bmatrix} = \begin{bmatrix} \text{row}_3\, I \\ \text{row}_2\, I \\ \text{row}_1\, I \end{bmatrix}$$

Then $EA = \begin{bmatrix} \text{row}_1\, EA \\ \text{row}_2\, EA \\ \text{row}_3\, EA \end{bmatrix} = \begin{bmatrix} (\text{row}_1 E)A \\ (\text{row}_2 E)A \\ (\text{row}_3 E)A \end{bmatrix} = \begin{bmatrix} (\text{row}_3 I)A \\ (\text{row}_2 I)A \\ (\text{row}_1\, I)A \end{bmatrix}$

$= \begin{bmatrix} \text{row}_3(IA) \\ \text{row}_2(IA) \\ \text{row}_1\,(IA) \end{bmatrix} = \begin{bmatrix} \text{row}_3 A \\ \text{row}_2 A \\ \text{row}_1\, A \end{bmatrix}$ So the effect of computing EA is

the same as performing the row operation "interchange rows 1 and 3" on A.

3) <u>Rescale a row by a nonzero constant c</u> Suppose, for example, that E is obtained by rescaling a row (let's say row 1) of I by a factor of k $(\neq 0)$.

$$E = \begin{bmatrix} \text{row}_1\, E \\ \text{row}_2\, E \\ \text{row}_3\, E \end{bmatrix} = \begin{bmatrix} k\,\text{row}_1\, I \\ \text{row}_2\, I \\ \text{row}_3\, I \end{bmatrix}$$

Then $EA = \begin{bmatrix} \text{row}_1\, EA \\ \text{row}_2\, EA \\ \text{row}_3\, EA \end{bmatrix} = \begin{bmatrix} (\text{row}_1 E)A \\ (\text{row}_2 E)A \\ (\text{row}_3 E)A \end{bmatrix} = \begin{bmatrix} (k\,\text{row}_1 I)A \\ (\text{row}_2 I)A \\ (\text{row}_1\, I)A \end{bmatrix}$

$= \begin{bmatrix} k\,\text{row}_1(IA) \\ \text{row}_2(IA) \\ (\text{row}_1\,(IA) \end{bmatrix} = \begin{bmatrix} k\,\text{row}_1 A \\ \text{row}_2\, A \\ (\text{row}_1\, A \end{bmatrix}$ So the effect of computing EA is

the same as performing the row operation "rescale row 1 by a factor of k" on A.