Homework Problem Supplement for the *Introduction to Diagonalization*
Read that material before attempting these problems.

How can we tell if a 2×2 *matrix* A *has any eigenvalues or eigenvectors?*

An eigenvector $\boldsymbol{x}$*, with eigenvalue* λ*, is (by definition) a nonzero solution to* $A\boldsymbol{x} = \lambda\boldsymbol{x}$*. This equation can be rewritten as*

$$A\boldsymbol{x} - \lambda\boldsymbol{x} = \mathbf{0}$$
$$(A - \lambda I)\boldsymbol{x} = \mathbf{0} \qquad \textit{where } I = I_2 = \textit{ the } 2 \times 2 \textit{ identity matrix}$$

This equation will have a nonzero solution for $\boldsymbol{x}$ *if and only if det*$(A - \lambda I) = 0$.

Given a specific matrix A*, det*$(A - \lambda I)$ *is an expression involving an "unknown"* λ*.*
If we find a solution λ *for the equation det*$(A - \lambda I) = 0$*, then, for that* λ*,* $A\boldsymbol{x} = \lambda\boldsymbol{x}$ *will have a nontrivial solution for* $\boldsymbol{x}$*. Each such solution* $\boldsymbol{x} \neq \mathbf{0}$ *is an eigenvector of* A *with eigenvalue* λ*.*

If, for example, $\lambda = -5$ *is a solution to det*$(A - \lambda I) = 0$*, then we know* -5 *is an eigenvalue because equation* $A\boldsymbol{x} = -5\boldsymbol{x}$ *must have nontrivial solutions for* $\boldsymbol{x}$*. These nontrivial solutions are the eigenvectors corresponding to eigenvalue* $\lambda = -5$*. We would find these eigenvectors by solving* $A\boldsymbol{x} = -5\boldsymbol{x}$ *. To do that, rewrite it as a homogeneous system*

$$A\boldsymbol{x} = -5\,I\boldsymbol{x}$$

$$(A + 5I\,)\boldsymbol{x} = \mathbf{0}$$

This seems a bit abstract until you actually think through the steps in a specific situation – so here goes.

a) Let $A = \begin{bmatrix} 1 & 12 \\ 1 & 2 \end{bmatrix}$. Write the matrix $(A - \lambda I)$, and evaluate $\det(A - \lambda I)$. *Your answer is in terms of an "unknown"* λ *.*

Then solve the equation $\det(A - \lambda I) = 0$ for λ .

For this problem, you should find two distinct solutions : $\lambda =$ *? and* $\lambda =$ *??.*

b) Separately, for each λ from part a): substitute λ in the equation $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ and find the solutions for $\boldsymbol{x}$ (the nonzero solutions are the eigenvectors for that eigenvalue λ .)

c) Is the matrix A diagonalizable? (*Look at Theorem 2 in the notes "Introduction to Diagonalization."*) If it is, what are the matrices P, D, P^{-1} for which $A = PDP^{-1}$?

2. Suppose $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Notice that $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Choose $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \}$ to be a new basis for $\mathbb{R}^2$

If $\boldsymbol{x}$ is in $\mathbb{R}^2$ and $[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$, what is $A\boldsymbol{x}$ **?**

(*Note: try to answer the question without actually multiplying out*

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ *to get the matrix* A .)

3. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

a) Describe the mapping $\boldsymbol{x} \mapsto A\boldsymbol{x}$ <u>geometrically</u>. Explain briefly how you know this. (How does A "move things around"? *Hint: Consider first what the mapping does to* $\boldsymbol{e_1}$, *then* $\boldsymbol{e_2}$. *What does the mapping do to the unit square?)*

b) A has <u>no</u> eigenvalues.

i) Explain why there are no eigenvalues using some <u>algebraic</u> calculations: can $A\boldsymbol{x} = \lambda\boldsymbol{x}$ ever be true if $\boldsymbol{x} \neq \mathbf{0}$?

ii) Explain why there are no eigenvalues using the geometry of your answer in part a)

c) Is A diagonalizable?