Inner Product Spaces

In $\mathbb{R}^n$, we have an inner product $\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{u}^T\boldsymbol{v}=u_1v_1+...+u_nv_n$. Another notation sometimes used is

$$\boldsymbol{<u,v>}\ =\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{u}^T\boldsymbol{v}=u_1v_1+...+u_nv_n$$

The inner product in $<u,v>$ in $\mathbb{R}^n$ has several essential properties (*see Theorem 1, p. 331*) that we have used repeatedly:

a) $<\boldsymbol{u},\boldsymbol{v}>\ =\ <\boldsymbol{v},\boldsymbol{u}>$
b) $<\boldsymbol{u}+\boldsymbol{v},\boldsymbol{w}>\ =\ <\boldsymbol{u},\boldsymbol{w}>\ +\ <\boldsymbol{v},\boldsymbol{w}>$
c) $<c\boldsymbol{u},\boldsymbol{v}>\ =c<\boldsymbol{u},\boldsymbol{v}>$
d) $<\boldsymbol{u},\boldsymbol{u}>\ \geq 0$ and $<\boldsymbol{u},\boldsymbol{u}>\ =0$ if and only if $\boldsymbol{u}=\mathbf{0}$.

We defined the "length" of a vector $\boldsymbol{u}$ by $\|\boldsymbol{u}\|\ =\sqrt{<\boldsymbol{u},\boldsymbol{u}>}$, and the distance between two vectors $\boldsymbol{u},\boldsymbol{v}$ as $||\,\boldsymbol{u}-\boldsymbol{v}\,||=\sqrt{<\boldsymbol{u}-\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}>}$

Earlier in the course, we used the essential properties of vectors in $\mathbb{R}^n$ as the starting point to define more general vector spaces V (*p. 190*). In the same spirit, we now use the properties a)-d) to describe the "essential properties" for an inner product in any vector space $-$ for a vector space V with real scalars: any rule that creates a scalar $<\boldsymbol{u},\boldsymbol{v}>$ for each pair of vectors $\boldsymbol{u},\boldsymbol{v}$ in V and satisfies a)-d) will be called an inner product in V. (*Properties a)-d) are modified slightly when complex scalars are allowed.*) A vector space V with an inner product defined is called an inner product space. Because such an inner product "acts just like" the inner product from $\mathbb{R}^n$, many of the theorems we proved about inner products for $\mathbb{R}^n$ remain true for inner products in other spaces. You can look at a basic introduction to this material in Section 6.7 of the textbook.

Here is a little more detail involving one specific example

Let $C[-\pi,\pi]$ be the vector space of all continuous real-valued functions defined on the interval $[-\pi,\pi]$: call it C, for short.

For vectors (functions) f,g in C, define an inner product by

$$<f,g>\ =\text{the number }\int_{-\pi}^{\pi}f(x)g(x)\,dx$$

Then $<f,g>$ satisfies all the essential properties a)-d) for an inner product listed above:

a) $<f,g>\ =\ <g,f>$
because $\int_{-\pi}^{\pi}f(x)g(x)dx=\int_{-\pi}^{\pi}g(x)f(x)\,dx$

b) $<f+g,h>\ =\ <f,h>\ +\ <g,h>$
because $\int_{-\pi}^{\pi}(f(x)+g(x))h(x)\,dx=\int_{-\pi}^{\pi}f(x)h(x)\,dx+\int_{-\pi}^{\pi}g(x)h(x)\,dx$

c) $<cf,g>\ =c<f,g>$
d) $<f,f>\ \geq 0$ and $<f,f>\ =0$ if and only if $f=\mathbf{0}$ ($=$ the constant function $\mathbf{0}$)
You should check c) and d). The last part of d) requires that you use the fact that functions f in C are continuous.

Continuing in parallel with our definitions in $\mathbb{R}^n$:

Define the norm (or "length") of f by $||f|| = \sqrt{< f, f >} = \sqrt{\int_{-\pi}^{\pi} f^2(x)\,dx}$

and the distance between f and g as

$$|| f - g || = \sqrt{\int_{-\pi}^{\pi} (f(x) - g(x))^2\,dx}$$

We say that f and g are orthogonal $(f \perp g)$ if $< f, g > = \int_{-\pi}^{\pi} f(x)g(x)\,dx = 0$

For example: on $[-\pi, \pi]$, we have sin $\perp$ cos are orthogonal because

$$< \mathit{sin}, \mathit{cos} > = \int_{-\pi}^{\pi} (\mathit{sin}\, x)(\mathit{cos}\, x)\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi} \mathit{sin}(2x)\,dx = -\tfrac{1}{2}\mathit{cos}(2x)|_{-\pi}^{\pi} = 0$$

Many of the techniques we developed using inner products still work. For example:

For a subspace W of C: we can define $W^{\perp} = \{f : < f, g > = 0 \text{ for all } g \text{ in } W\}$

If W is a subspace of C with an orthogonal basis* $\{g_1, ..., g_n\}$ and $f \in C$

then we can uniquely write $f = \widehat{f} + g$ where $\widehat{f} \in W$ and $g \in W^{\perp}$.

$\widehat{f}$ is called the projection of f on W and $\widehat{f}$ is given by the formula

$$\widehat{f} = \tfrac{<f, g_1>}{<g_1, g_1>} g_1 + ... + \tfrac{<f, g_n>}{<g_n, g_n>} g_n$$

Then $\widehat{f}$ is the function in W closest to f, that is the function in W for which

$$\| f - \widehat{f} \| < \| f - g \| \text{ for all } g \text{ in } W \text{ different from } \widehat{f}$$

If $\{f_1, f_2, ..., f_n\}$ is a basis for a subspace W of C, we can convert the basis into an orthogonal basis using the same Gram Schmidt process formulas.

Note: Unlike $\mathbb{R}^n$, C is not finite dimensional. But for the results just listed, that doesn't matter. What does matter is that the subspace W is finite dimensional.

A sample calculation with polynomials in C

Let W be the subspace of polynomials on $[-\pi, \pi]$ with degree ≤ 5 :

$$W = \text{Span}\{1, x, x^2, x^3, x^4, x^5\}$$

Find the polynomial in W closest to the function sin. (*Watch how the steps parallel what we'd do in $\mathbb{R}^n$: Matlab will handle the details for us*).

The polynomial we want is $q = \widehat{\sin} = \text{proj}_W \sin$. This polynomial is the closest in W to the function sin in the sense of the distance we defined:

The approximation error $||\, q \,-\, \sin \,|| = \left(\int_{-\pi}^{\pi} |\, q(x) - \sin x|^2 \, dx \right)^{1/2}$ is smaller than

$$||\, p \,-\, \sin \,|| = \left(\int_{-\pi}^{\pi} |\, p(x) - \sin x|^2 \, dx \right)^{1/2} \text{ for any other } p \in W$$

In that sense, q is the "best available approximation in W for sin."

It's easy to compute $\text{proj}_U \sin$ if we choose an orthonormal basis for W, so we convert the
standard basis $\{v_1, v_2, ..., v_6\} = \{1, x, x^2, ..., x^5\}$ for W
into an orthonormal basis we'll call $\{e_1, e_2, e_3, ..., e_6\}$
using the Gram-Schmidt Process (with normalization at each step).

Note: the integrations below were done using Matlab. Notice that every integration needed to find the e_i's is very easy, but that the constants that arise are messy and pile up fast; they can easily lead to errors when the computation is done by hand. Try to compute at least e_1, e_2, e_3 for yourself (with or without computer assistance) to be sure you understand what's going on. The steps are the same as for the usual Gram Schmidt process in $\mathbb{R}^n$.

We start with the first basis vector $v_1 = 1$. But v_1 is not a unit vector in C because
$||v_1||^2 = ||1||^2 = \int_{-\pi}^{\pi} 1 \cdot 1 \, dx = 2\pi$. So we normalize and use

$$e_1 = \frac{v_1}{||\, v_1 \,||} = \frac{1}{||1||} = \frac{1}{\sqrt{2\pi}}$$

For $j = 2, ..., 6$ in turn we use the Gram Schmidt formula, normalizing at each step to get a unit vector:

$$e_j = \frac{v_j - <v_j,e_1>e_1 - <v_j,e_2>e_2 - ... - <v_j,e_{j-1}>e_{j-1}}{\|\, v_j - <v_j,e_1>e_1 - <v_j,e_2>e_2 - ... - <v_j,e_{j-1}>e_{j-1} \,\|}$$

So $$e_2 = \frac{v_2 - <v_2,e_1>e_1}{||v_2 - <v_2,e_1>e_1||} = \frac{x - \left(\int_{-\pi}^{\pi} x \cdot \frac{1}{\sqrt{2\pi}} \, dx\right) \cdot \frac{1}{\sqrt{2\pi}}}{||x - \left(\int_{-\pi}^{\pi} x \cdot \frac{1}{\sqrt{2\pi}} \, dx\right) \cdot \frac{1}{\sqrt{2\pi}}||}$$

$$= \frac{x}{||x||} \qquad (\text{since } \int_{-\pi}^{\pi} x \, dx = 0)$$

$$= \frac{x}{\left(\int_{-\pi}^{\pi} x \cdot x \, dx\right)^{1/2}} = \frac{x}{\frac{1}{3}\,\pi (6\pi)^{1/2}} = \frac{\sqrt{6}\, x}{2\pi \sqrt{\pi}}$$

Then e_3

$$= \frac{v_3 - <v_3,e_1>e_1 - <v_3,e_2>e_2}{||\, v_3 - <v_3,e_1>e_1 - <v_3,e_2>e_2 \,||} = \frac{x^2 - (\int_{-\pi}^{\pi} x^2 \cdot e_1 \, dx) e_1 - (\int_{-\pi}^{\pi} x^2 \cdot e_2 \, dx) e_2}{||\, x^2 - (\int_{-\pi}^{\pi} x^2 \cdot e_1 \, dx) e_1 - (\int_{-\pi}^{\pi} x^2 \cdot e_2 \, dx) e_2 \,||}$$

$$= \frac{x^2 - (\int_{-\pi}^{\pi} x^2 \cdot \frac{1}{\sqrt{2\pi}} \, dx) \cdot \frac{1}{\sqrt{2\pi}} - (\int_{-\pi}^{\pi} x^2 \cdot \frac{\sqrt{6}\, x}{2\pi \sqrt{\pi}} \, dx) \cdot \frac{\sqrt{6}\, x}{2\pi \sqrt{\pi}}}{||\, x^2 - (\int_{-\pi}^{\pi} x^2 \cdot \frac{1}{\sqrt{2\pi}} \, dx) \cdot \frac{1}{\sqrt{2\pi}} - (\int_{-\pi}^{\pi} x^2 \cdot \frac{\sqrt{6}\, x}{2\pi \sqrt{\pi}} \, dx) \cdot \frac{\sqrt{6}\, x}{2\pi \sqrt{\pi}} \,||}$$

Notice that each integration requires no calculus harder than $\int x^n \, dx$.
But already the constants are becoming a headache.

$$= \quad \ldots \quad = \frac{1}{8} \frac{\sqrt{8}\, \sqrt{45}\, (x^2 - \frac{1}{3}\pi^2)}{\sqrt{\pi^5}}$$

Continuing in this way gives:

$$e_4 = \quad \ldots \quad = \frac{1}{8} \frac{\sqrt{175}\, \sqrt{8}\, (x^3 - \frac{3}{5}\pi^2 x)}{\sqrt{\pi^7}}$$

$$e_5 = \quad \ldots \quad = \frac{1}{128} \frac{\sqrt{11025}\, \sqrt{128} \left(x^4 - \frac{1}{5}\pi^4 - \frac{6}{7}\pi^2 (x^2 - \frac{1}{3}\pi^2) \right)}{\sqrt{\pi^9}}, \text{ and finally}$$

$$e_6 = \quad \ldots \quad = \frac{1}{128} \frac{\sqrt{128}\, \sqrt{43659} \left(x^5 - \frac{3}{7}\pi^4 x - \frac{10}{9}\pi^2 (x^3 - \frac{3}{5}\pi^2 x) \right)}{\sqrt{\pi^{11}}}$$

Then $e_1, \ldots, e_6$ are orthogonal polynomials of degree ≤ 5; we use them as our orthogonal basis for W. With them, we can use the projection formula to compute

$$q(x) = \text{Proj}_W \sin x = \; < \sin x, e_1 > e_1 + \; < \sin x, e_2 > e_2 + \ldots + \; < \sin x, e_6 > e_6$$

$$= (\int_{-\pi}^{\pi} (\sin x) e_1 \, dx) e_1 + (\int_{-\pi}^{\pi} (\sin x) e_2 \, dx) e_2 + \ldots + (\int_{-\pi}^{\pi} (\sin x) e_6 \, dx) e_6$$

$$= (\int_{-\pi}^{\pi} (\sin x) \frac{1}{\sqrt{2\pi}} \, dx) \frac{1}{\sqrt{2\pi}} + (\int_{-\pi}^{\pi} (\sin x) \frac{\sqrt{6}\, x}{2\pi \sqrt{\pi}} \, dx) \cdot \frac{\sqrt{6}\, x}{2\pi \sqrt{\pi}}$$

$$+ (\int_{-\pi}^{\pi} (\sin x) \frac{1}{8} \frac{\sqrt{8}\, \sqrt{45}\, (x^2 - \frac{1}{3}\pi^2)}{\sqrt{\pi^5}} \, dx) \cdot \frac{1}{8} \frac{\sqrt{8}\, \sqrt{45}\, (x^2 - \frac{1}{3}\pi^2)}{\sqrt{\pi^5}}$$

$+ \ldots +$ three more terms corresponding to e_4, e_5, and e_6

(*Notice that the integrations involved are now more challenging because they involve terms like $\int x^n \sin x \, dx$. But they are manageable, with enough patience, using integration by parts.*)

When all the smoke clears, Matlab gives

$$q(x) = \frac{21}{8\pi^{10}} \big((33\pi^4 - 3465\pi^2 + 31185)x^5 + (3750\pi^4 - 30\pi^6 - 34650\pi^2)x^3 + (5\pi^8 - 765\pi^6 + 7425\pi^4)x)\big) \quad (*)$$

If we convert the exact coefficients in (*) to approximate decimal coefficients we have

$$q(x) \approx 0.98786213557467x - 0.15527141063343\, x^3 + 0.00564311797635\, x^5$$

In the sense of the distance $|| \ ||$ that we defined in C, q is the closest polynomial with degree ≤ 5 to the function sin. (*Remember from the definition of* C : *all this is happening over the interval* $[-\pi, \pi]$.)

For comparison: there is a 5th degree polynomial approximation for $\sin x$ that is better known $-$ namely, the 5th degree Taylor polynomial,

$T_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^{(v)}(x)}{5!}x^5.$
For $f(x) = \sin x,$ this gives

$$\sin x \approx T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

With approximate decimal coefficients,

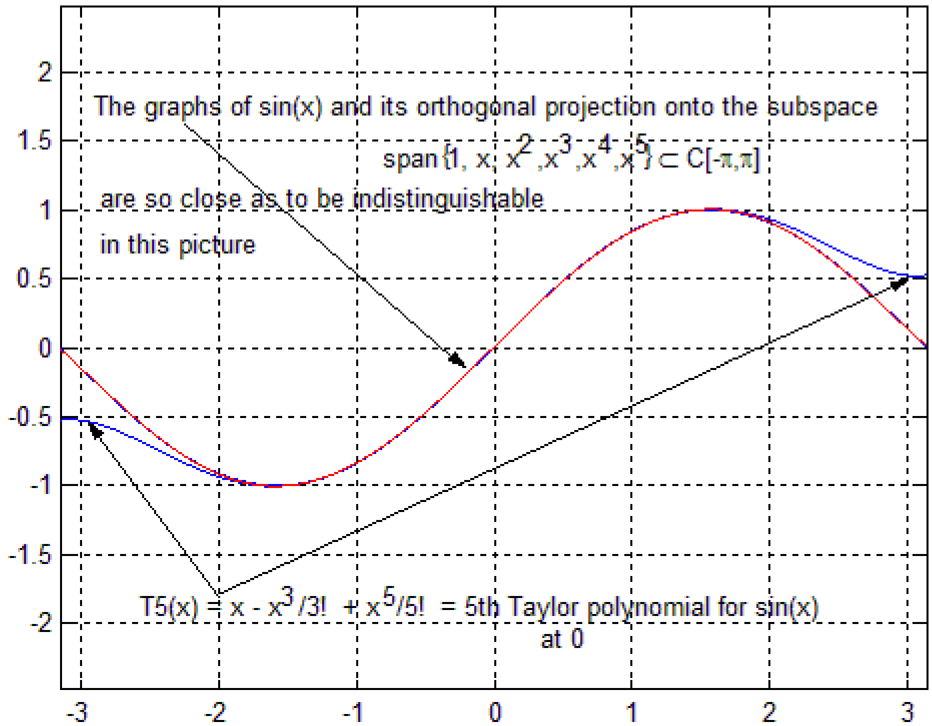
$$T_5(x) = x - 0.16666666666667x^3 + 0.00833333333333x^5$$

Because $T_5(x)$ is constructed using derivatives of f at 0, it turns to be the better approximation for $\sin x$ near 0, but it produces a larger and larger error as x gets further and further from 0.

We can see this in the figure on the next page.

The figure below shows the graphs of $\sin x$, $T_5(x)$ and $q(x)$ on the interval $[-\pi, \pi]$.

<u>Across the whole interval</u> $[-\pi, \pi]$: the graphs of $\sin x$ and $q(x)$ are so close that they visually coincide at the scale of this graph: you can see the difference between them. Near 0, you also can't see the difference between $T_5(x)$ and $\sin x$, but that difference becomes clear in the picture as you move further from 0 and closer to the endpoints $\pm\pi$.



The table on the following page illustrates three things (the third is not clear from the graphs above):

i) As we move away from 0 toward $\pm\pi$, the approximation to $\sin x$ using $T_5(x)$ is not as good as the $q(x)$ approximation

ii) Linear algebra gives us the $q(x)$ approximation. It has the advantage that it gives us a good approximation for $\sin x$ <u>over the whole interval</u> $[-\pi, \pi]$.

iii) The Taylor polynomial $T_5(x)$ is actually a <u>better</u> approximation than $q(x)$ to $\sin x$ when x is near 0 (*in some sense,* $T_5(x)$ *is actually the best of all 5th degree polynomials to approximate* $\sin x$ <u>*near*</u> 0).

All table values are rounded to 4 significant digits
so, for example, 0.0000 is not exactly 0

x	$\sin x$	$T_5(x)$	$q(x)$	$\lvert\text{Error}\rvert = \lvert \sin x - T_5(x) \rvert$		$\lvert\text{Error}\rvert = \lvert \sin x - q(x) \rvert$	
-3.1416	-0.0000	-0.5240	-0.0160	0.5240	← **large** T_5	0.0160	← **smallish** $\boldsymbol{q}$
-2.9416	-0.1987	-0.5347	-0.1966	0.3361	← **|error|**	0.0021	← **|error|**
-2.7416	-0.3894	-0.5979	-0.3827	0.2084	← **near** $-\boldsymbol{\pi}$	0.0067	← **over**
-2.5416	-0.5646	-0.6891	-0.5600	0.1244	⋮	0.0047	← **the whole**
-2.3416	-0.7174	-0.7884	-0.7169	0.0710		0.0005	← **interval**
-2.1416	-0.8415	-0.8800	-0.8447	0.0385		0.0032	⋮
-1.9416	-0.9320	-0.9516	-0.9372	0.0196		0.0052	
-1.7416	-0.9854	-0.9947	-0.9906	0.0092		0.0052	↕
-1.5416	-0.9996	-1.0035	-1.0032	0.0040		0.0036	
-1.3416	-0.9738	-0.9754	-0.9749	0.0015		0.0011	
-1.1416	-0.9093	-0.9098	-0.9077	0.0005		0.0016	
-0.9416	-0.8085	-0.8086	-0.8047	0.0001		0.0038	
-0.5416	-0.5155	-0.5155	-0.5106	0.0000	← **very**	0.0049	← **but |error|**
-0.3416	-0.3350	-0.3350	-0.3313	0.0000	← **small**	0.0037	← **for** $\boldsymbol{q}$
-0.1416	-0.1411	-0.1411	-0.1394	0.0000	← **|error|**	0.0017	← **near 0**
0.0584	0.0584	0.0584	0.0577	0.0000	← **near 0**	0.0007	← **is larger**
0.2584	0.2555	0.2555	0.2526	0.0000	← **using**	0.0029	← **than for the**
0.4584	0.4425	0.4425	0.4380	0.0000	← $\boldsymbol{T_5}$	0.0045	← $\mathbf{T_5}$
0.6584	0.6119	0.6119	0.6068	0.0000	← **approx**	0.0051	← **approx**
0.8584	0.7568	0.7569	0.7524	0.0001	⋮	0.0044	⋮
1.0584	0.8716	0.8719	0.8690	0.0003		0.0026	
1.2584	0.9516	0.9526	0.9515	0.0010		0.0001	
1.4584	0.9937	0.9964	0.9963	0.0027		0.0026	
1.6584	0.9962	1.0028	1.0009	0.0066		0.0047	
1.8584	0.9589	0.9734	0.9644	0.0145		0.0054	↕
2.0584	0.8835	0.9128	0.8877	0.0293		0.0043	
2.2584	0.7728	0.8282	0.7740	0.0554		0.0012	
2.4584	0.6313	0.7304	0.6283	0.0991		0.0030	← **smallish** $\boldsymbol{q}$
2.6584	0.4646	0.6336	0.4583	0.1690	⋮	0.0063	← **|error|**
2.8584	0.2794	0.5561	0.2742	0.2767	← **large** $\boldsymbol{T_5}$	0.0052	← **over**
3.0584	0.0831	0.5204	0.0894	0.4373	← **|error|**	0.0063	← **the whole**
3.1416	9.0000	0.5240	0.0160	0.5240	← **near** $\boldsymbol{\pi}$	0.0160	← **interval**

One more example, without many details

We are still working in the vector space C of continuous functions on the interval $[-\pi, \pi]$.

Pink an n and consider the subspace

$$W = \text{Span}\,\{1,\ \sin x,\ \sin 2x, ...,\ \sin nx,\ \cos x,\ ...,\ \cos nx\}$$

A function in W is a linear combination of the basis elements:

$$(*)\quad \boldsymbol{a_0 + a_1 \cos x + ... + a_n \cos nx + b_1 \sin x + ... + b_n \sin nx} \qquad (*)$$

You can check that these functions form an orthogonal basis for W :

For example, to show $\sin 2x \perp \cos 3x$:

$$< \sin 2x, \cos 3x > \ = \int_{-\pi}^{\pi} (\sin 2x)(\cos 3x)\, dx$$

Since	$\sin(\alpha + \beta)$	$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$
and	$\sin(\alpha - \beta)$	$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$
then	$\sin(\alpha + \beta) + \sin(\alpha - \beta)$	$= 2\sin\alpha\cos\beta$

If we let $\alpha = 2x$ and $\beta = 3x$, we

	$\sin(5x) + \sin(-x)$	$= 2\sin(2x)\cos(3x)$
so	$\frac{1}{2}(\sin(5x) - \sin(x))$	$= \sin(2x)\cos(3x)$

Therefore $\int_{-\pi}^{\pi} (\sin 2x)(\cos 3x)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(5x) - \sin(x)\, dx$

$$= \tfrac{1}{2}\left(-\tfrac{1}{5}\cos 5x + \cos x\, dx\right)\Big|_{-\pi}^{\pi} = 0$$

If f is any function in C, we can compute $\text{proj}_W\, f = \widehat{f} =$ the function in W closest to f, $\widehat{f}$ is a function that looks like $(*)$

$$\widehat{=} \ = \frac{<f(x),1>}{<1,1>}1 + \frac{<f(x),\cos x>}{<\cos x,\cos x>}\cos x + ... + \frac{<f(x),\cos nx>}{<\cos nx,\cos nx>}\cos nx$$

$$+ \frac{<f(x),\sin x>}{<\sin x,\sin x>}\sin x + ... + \frac{<f(x),\sin nx>}{<\sin nx,\sin nx>}\sin nx$$

For the denominators, we can calculate

$$< 1, 1 > \ = \int_{-\pi}^{\pi} 1 \cdot 1\, dx = 2\pi$$

$$< \cos kx, \cos kx> = \int_{-\pi}^{\pi} \cos^2 kx\, dx = \pi \qquad \textit{(why?)}$$

$$< \sin kx, \sin kx> = \int_{-\pi}^{\pi} \sin^2 kx\, dx = \pi \qquad \textit{(why?)}$$

Therefore $\text{proj}_W\, f = \widehat{f}$

$$= \tfrac{1}{2}\left(\tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot 1\,dx\right) 1$$

$$+\ \tfrac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\cdot \cos x\ dx\right)\cos x + \ldots +\ \tfrac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\cdot \cos nx\,dx\right)\cos nx$$

$$+\ \tfrac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\cdot \sin x\ dx\right)\sin x + \ldots +\ \tfrac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\cdot \sin nx\,dx\right)\sin nx$$

$$= \tfrac{a_0}{2} + a_1\cos x + \ldots + a_n\cos nx + b_1\sin x + \ldots + b_n\sin nx$$

So $\text{proj}_W\, f = \widehat{f}\ = \tfrac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx + \sum_{k=1}^{n}\sin kx$ (*)

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot 1\,dx$

and $\begin{cases} a_k = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\cdot\cos kx\,dx\right) & \text{for } 1 < k \le n \\ b_k = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\cdot\sin nx\,dx\right) & \text{for } 1 < k \le n \end{cases}$

The coefficients used in writing $\text{proj}_W\, f$ are called the Fourier coefficients of f

and the "trigonometric series" (*) is the n^{th} Fourier approximation for f.

If $n \to \infty$, *then* $\|\, f - \text{proj}_W f \,\| \to 0$, *that is, the approximation error* $\to 0$ (*as measured by our distance function* $\|\, f - \text{proj}_W f \,\|$. (*This is called "mean square convergence."*)

"Mean square convergence" is ***not*** *the same as saying:*

for each $x \in [-\pi, \pi]$, $\widehat{f}(x) \to f(x)$ (*this is called pointwise convergence*)
It's a much harder problem to determine for which x*'s this is true.*

Pointwise convergence is true, for example, at any point x *where* f *is differentiable.*