

System of Linear Equations

variables (“unknowns”)

$$x_1, \quad x_2, \quad \dots, \quad x_n$$
$$\downarrow \qquad \downarrow \qquad \qquad \downarrow$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

↑

↑

↑

constant coefficients (real or complex numbers)

Linear system of equations

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
 &\vdots \\
 a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n &= b_i \\
 &\vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
 \end{aligned}$$

“Record essential information” using the

$$\begin{array}{ccc}
 \leftarrow & \text{augmented matrix} & \rightarrow \\
 \\
 \left[\begin{array}{cccc|c}
 a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
 a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
 \hline
 a_{i1} & a_{i2} & \dots & a_{in} & b_i \\
 \hline
 a_{m1} & a_{m2} & \dots & a_{mn} & b_m
 \end{array} \right] \\
 \\
 \leftarrow & \text{coefficient matrix} & \rightarrow
 \end{array}$$

Elementary Row Operations (EROs)

(for a system of linear equations, or for any matrix)

- 1) interchange any two rows (= row swap)
- 2) multiply any row by a nonzero constant c (= row rescale)
- 3) add a multiple of one row to another row (= row replacement)

Two matrices (or, systems of equations) are **row equivalent** if you can change one into the other by using a sequence of EROs

Important: Each ERO operation is reversible – that is, can be “undone” by using another ERO.

$\left\{ \begin{array}{l} \text{Swap: interchange row 1 (R1) and row 2 (R2)} \\ \text{is reversed by the ERO} \\ \text{????} \end{array} \right.$

$\left\{ \begin{array}{l} \text{Rescale: multiply R3 by 5} \\ \text{is reversed by the ERO} \\ \text{????} \end{array} \right.$

$\left\{ \begin{array}{l} \text{Replace: add 3*R1 to R3} \\ \text{is reversed by the ERO} \\ \text{????} \end{array} \right.$

Examples

$$\begin{cases} 2x_1 - x_2 = -3 \\ x_1 - x_2 = 1 \end{cases} \sim \begin{cases} x_1 - x_2 = 1 \\ 2x_1 - x_2 = -3 \end{cases} \sim \begin{cases} x_1 - x_2 = 1 \\ 0 + x_2 = -5 \end{cases}$$

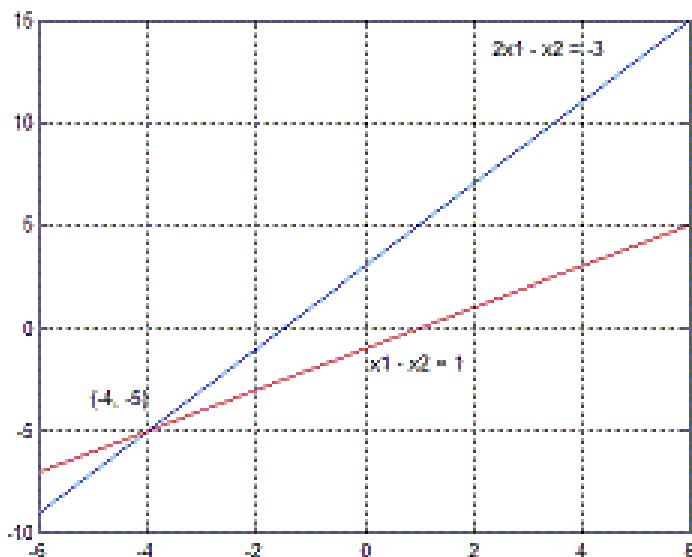
$$\sim \begin{cases} x_1 + 0 = -4 \\ 0 + x_2 = -5 \end{cases}$$

Written as (augmented) matrices:

$$\left[\begin{array}{ccc} 2 & -1 & -3 \\ 1 & -1 & 1 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & -1 & 1 \\ 2 & -1 & -3 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 1 & -5 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc} 1 & 0 & -4 \\ 0 & 1 & -5 \end{array} \right]$$

Solutions $x_1 = -4, x_2 = -5$



Most important questions about any linear system of equations

1) Is there a solution for the system?

“yes” \leftrightarrow consistent
system

“no” \leftrightarrow inconsistent
system

2) If there is a solution (= consistent system)

a) is there only one solution
(*solution is unique*)

or

b) is there more than one solution?

*Fact: For a linear system of equations, if
there is more than one solution, then
it turns out that there must be an
infinite number of solutions*

Example

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 + 6x_3 &= 12 \end{aligned}$$

$$\begin{array}{c} \text{Aug. Matrix} \\ \downarrow \\ \left[\begin{array}{ccccc} 1 & -1 & 1 & 0 \\ 2 & 3 & 6 & 12 \end{array} \right] \end{array} \sim \begin{array}{c} \text{row equivalent to} \\ \downarrow \\ \left[\begin{array}{cccc} 1 & -1 & 1 & 0 \\ 0 & 5 & 4 & 12 \end{array} \right] \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 4/5 & 12/5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 9/5 & 12/5 \\ 0 & 1 & 4/5 & 12/5 \end{bmatrix}$$

↑ ↑ ↑ ↑

$x_1 \ x_2 \ x_3$ constants

SO

$$x_1 = 12/5 - \frac{9}{5}x_3 \quad \text{and}$$

$$x_2 = 12/5 - 4/5x_3$$

where x_3 is “free” (can have any value)

Example

$$\begin{aligned}x_1 - 3x_2 + 4x_3 &= -4 \\3x_1 - 7x_2 + 7x_3 &= -8 \\-4x_1 + 6x_2 - x_3 &= 7\end{aligned}$$

$$\left[\begin{array}{cccc} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{array} \right] \sim \left[\begin{array}{cccc} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ -4 & 6 & -1 & 7 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{array} \right] \sim \left[\begin{array}{cccc} 1 & -3 & 4 & -4 \\ 0 & 1 & -5/2 & 2 \\ 0 & -6 & 15 & -9 \end{array} \right]$$

Last row says: $0x_1 + 0x_2 + 0x_3 = 3$

IMPOSSIBLE: SYSTEM IS INCONSISTENT

Exercise

Suppose we start with the augmented matrix for a system of linear equations and using EROs we get to the matrix shown below.

How many equations and variables in the original system?

What can you say about solutions of the system?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Example

$$\begin{array}{rcl}
 x_1 + x_2 + x_3 & = & 1 \\
 2x_1 + 3x_2 - x_3 & = & -4 \\
 - & x_1 + 2x_2 + x_3 & = 3
 \end{array}
 \quad \text{In matrix notation:}$$

$$\text{Augmented matrix} = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 3 & -1 & -4 \\ -1 & 2 & 1 & 3 \end{array} \right]$$

$$\sim (\text{What ERO?}) \quad \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & -6 \\ -1 & 2 & 1 & 3 \end{array} \right]$$

$$\sim (\text{What ERO?}) \quad \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & -6 \\ 0 & 3 & 2 & 4 \end{array} \right]$$

$$\sim (\text{What ERO?}) \quad \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 11 & 22 \end{array} \right]$$

$$\sim (\text{What ERO?}) \quad \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\sim (\text{What ERO?}) \quad \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\sim (\text{What ERO?}) \quad \left[\begin{array}{cccc} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\sim (\text{What ERO?}) \quad \left[\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

which gives that:

$$\left\{ \begin{array}{lcl} x_1 & = & ? \\ x_2 & = & ? \\ x_3 & = & ? \end{array} \right.$$

Example

$$\begin{aligned}x_1 - 3x_2 + 4x_3 &= -4 \\3x_1 - 7x_2 + 7x_3 &= -8 \\-4x_1 + 6x_2 - x_3 &= 7\end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{array} \right]$$

$$\sim (\text{What ERO?}) \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ -4 & 6 & -1 & 7 \end{array} \right]$$

$$\sim (\text{What ERO?}) \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{array} \right]$$

$$\sim (\text{What ERO?}) \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 1 & -5/2 & 2 \\ 0 & -6 & 15 & -9 \end{array} \right]$$

$$\sim (\text{What ERO?}) \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 1 & -5/2 & 2 \\ 0 & 0 & \mathbf{0} & \mathbf{3} \end{array} \right] \text{ so ... ?}$$