Orthogonal Decomposition Theorem

Suppose $\boldsymbol{y}$ is in $\mathbb{R}^n$ and that W is a subspace of $\mathbb{R}^n$.

Let $\{\boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_p}\}$ be an orthogonal basis for W.

We can write $\boldsymbol{y} = \widehat{\boldsymbol{y}} + \boldsymbol{z}$ where $\begin{cases} \widehat{y} \in W \\ z \in W^{\perp} \end{cases}$ and using

$$\begin{cases} \widehat{\boldsymbol{y}} = \frac{\boldsymbol{y}\cdot\boldsymbol{u_1}}{\boldsymbol{u_1}\cdot\boldsymbol{u_1}}\boldsymbol{u_1} + ... + \frac{\boldsymbol{y}\cdot\boldsymbol{u_i}}{\boldsymbol{u_i}\cdot\boldsymbol{u_i}}\boldsymbol{u_i} + ... + \frac{\boldsymbol{y}\cdot\boldsymbol{u_p}}{\boldsymbol{u_n}\cdot\boldsymbol{u_p}}\boldsymbol{u_p} \\ \quad = \text{proj}_{\boldsymbol{u_1}}\,\boldsymbol{y} + ... + \text{proj}_{\boldsymbol{u_1}}\,\boldsymbol{y} + ,,, + \text{proj}_{\boldsymbol{u_1}}\,\boldsymbol{y} \\ \\ \boldsymbol{z} = \boldsymbol{y} - \widehat{\boldsymbol{y}} \end{cases}$$

$\widehat{\boldsymbol{y}} = \text{proj}_W\,\boldsymbol{y} =$ projection of $\boldsymbol{y}$ on W
$\boldsymbol{z} = \boldsymbol{y} - \widehat{\boldsymbol{y}} \quad =$ component of $\boldsymbol{y}$ orthogonal to W

$\widehat{\boldsymbol{y}}$ and $\boldsymbol{z}$ are unique

Example Find the Least Squares Solution of

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 3 \\ -1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(*an inconsistent system*)

Easy because A happens to have orthogonal, so it's easy to find $\widehat{\boldsymbol{b}}$.

Let $W = \text{Col}\, A$. The columns of $A = [\boldsymbol{u_1}\ \boldsymbol{u_2}]$ are an orthogonal basis for W. Orthogonal decomposition theorem gives:

$$\widehat{\boldsymbol{b}} = \frac{\boldsymbol{b}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}\boldsymbol{u_1} + \frac{\boldsymbol{b}\cdot\boldsymbol{u}_2}{\boldsymbol{u}_2\cdot\boldsymbol{u}_2}\boldsymbol{u_2}$$

$$= \frac{0}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{13}\begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{6}{13} \\ -\frac{9}{13} \\ 0 \end{bmatrix}$$

Then just solve $A\boldsymbol{x} = \widehat{\boldsymbol{b}}$ directly:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 3 \\ -1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ -\frac{6}{13} \\ -\frac{9}{13} \\ 0 \end{bmatrix}$$

But the work for this is already done! in the solution

$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, x_1 and x_2 are the weights needed to

write $\begin{bmatrix} 0 \\ -\frac{6}{13} \\ -\frac{9}{13} \\ 0 \end{bmatrix}$ as a linear combination of the

columns of A – and we already know those weights:

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ -\frac{3}{13} \end{bmatrix} = \begin{bmatrix} \frac{\boldsymbol{b}\cdot\boldsymbol{u_1}}{u_1\cdot u_1} \\ \frac{\boldsymbol{b}\cdot\boldsymbol{u_2}}{u_2\cdot u_2} \end{bmatrix}.$$

For $A\boldsymbol{x} = \boldsymbol{b}$, the least squares solution is

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} 0 \\ -\frac{3}{13} \end{bmatrix}$$

The least square error here is

$$|| A\,\widehat{\boldsymbol{x}} \; - \boldsymbol{b} \,|| = \; || \,\widehat{\boldsymbol{b}} \; - \boldsymbol{b} \,||$$

$$= || \begin{bmatrix} 0 \\ -\frac{6}{13} \\ -\frac{9}{13} \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} ||$$

$$= \sqrt{(0-1)^2 + (-\tfrac{16}{13} - 0)^2 + (-\tfrac{9}{13} - 1)^2 + (0-1)^2}$$

$$\approx 2.5256$$

For any other $\boldsymbol{x}$, $||A\boldsymbol{x} - \boldsymbol{b}|| > 2.5256$

(" $>$ " here rather than " $\geq$ " because,

having solved, we know there are no <u>other</u> least squares solutions, that is, in this problem $\widehat{\boldsymbol{x}}$ is unique.

Example Find least squares solutions for $A\boldsymbol{x} = \boldsymbol{b}$ (*inconsistent*)

$$\begin{bmatrix} -5 & 2 & -3 & 2 & -1 \\ 1 & 1 & 2 & 1 & 0 \\ 3 & 1 & 4 & 1 & 1 \\ -1 & 1 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 3 \\ 6 \\ 2 \end{bmatrix}$$

Solve $A^T A\,\boldsymbol{x} = A^T \boldsymbol{b}$

$$\begin{bmatrix} 36 & -7 & 29 & -7 & 8 \\ -7 & 7 & 0 & 7 & -1 \\ 29 & 0 & 29 & 0 & 7 \\ -7 & 7 & 0 & 7 & -1 \\ 8 & -1 & 7 & -1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 19 \\ 11 \\ 30 \\ 11 \\ 6 \end{bmatrix}$$

The augmented matrix row reduces to

$$\begin{bmatrix} \mathbf{1} & 0 & 1 & 0 & 0 & \frac{19}{34} \\ 0 & \mathbf{1} & 1 & 1 & 0 & \frac{41}{17} \\ 0 & 0 & 0 & 0 & \mathbf{1} & \frac{67}{34} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 2 free variables, $\boldsymbol{x_3}$ and $\boldsymbol{x_4}$**.** The general solution is

$$\widehat{\boldsymbol{x}} = \boldsymbol{x} = \begin{bmatrix} \frac{19}{34} - x_3 \\ \frac{41}{17} - x_3 - x_4 \\ x_3 \\ x_4 \\ \frac{67}{34} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{19}{34} \\ \frac{41}{17} \\ 0 \\ 0 \\ \frac{67}{34} \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

All of these solutions are “least squares solutions.” If we choose any two of them, say $\widehat{\boldsymbol{x}}_1$ and $\widehat{\boldsymbol{x}}_2$, then

$A\widehat{\boldsymbol{x}}_1 = \widehat{\boldsymbol{b}}$ and $A\widehat{\boldsymbol{x}}_2 = \widehat{\boldsymbol{b}}$

so $\| A\widehat{\boldsymbol{x}}_1 - \boldsymbol{b} \| = \| A\widehat{\boldsymbol{x}}_2 - \boldsymbol{b} \|$

$= \| \widehat{\boldsymbol{b}} - \boldsymbol{b} \|$

$=$ least square error

<u>If there are infinitely many least squares solutions, then they all produce the same least square error</u>.

To compute least square error, pick <u>any</u> of the leeast squares

solutions, say with :$x_3 = 1,\ x_4 = 0$ and get one least squares

solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} \frac{19}{34} \\ \frac{41}{17} \\ 0 \\ 0 \\ \frac{67}{34} \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{15}{34} \\ \frac{24}{17} \\ 1 \\ 0 \\ \frac{67}{34} \end{bmatrix}$

Then $$A \begin{bmatrix} -\frac{15}{34} \\ \frac{24}{17} \\ 1 \\ 0 \\ \frac{67}{34} \end{bmatrix} = \begin{bmatrix} \frac{1}{17} \\ \frac{101}{34} \\ \frac{103}{17} \\ \frac{63}{34} \end{bmatrix} = \widehat{\boldsymbol{b}}$$

and the least squares error can be computed:

$$|| A\widehat{\boldsymbol{x}} - \boldsymbol{b} || = || \widehat{\boldsymbol{b}} - \boldsymbol{b} ||$$

$$= || \begin{bmatrix} \frac{1}{17} \\ \frac{101}{34} \\ \frac{103}{17} \\ \frac{63}{34} \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ 6 \\ 2 \end{bmatrix} || = \frac{1}{\sqrt{34}}$$