Homework 2, 9a)

Prove that if x and y are positive real numbers, then $\frac{x+y}{2} \geq \sqrt{xy}$. Where in the proof is it essential that x and y be positive?

(*Note before we start: "positive" really* <u>*isn't*</u> *necessary* ($x, y > 0$). *What* <u>*is*</u> *necessary is that* x *and* y *be* <u>*nonnegative*</u> $(x, y \geq 0)$

<u>Comments</u>

A proof is supposed to be a logical argument that <u>uses</u> already known mathematical facts and any given hypotheses to argue that a certain conclusion (***) must follow.

For a "direct proof" this means that <u>either the proof</u>:

1) <u>starts</u> with known facts or hypotheses of the theorem and <u>from these to</u> the conclusion (where the proofs <u>ends</u>).

We know that ...
So ...
and we are assuming that ...
So ...
$\vdots$

Therefore *** is true. •

<u>or the proof:</u>

2) contains a sequence of steps, all <u>equivalent</u> ($\Leftrightarrow$) to each other (with justification for the equivalence where needed) in which

a) the first or last step is a true statement (or hypothesis, which is assumed to be true for the theorem) , and

b) the last or first step is the desired conclusion

(*which means that a true statement is* <u>*equivalent*</u> *to the statement *** that you want to prove, and therefore *** must also be true.*)

OVER →

WRONG: <u>NOT</u> A PROOF

$\frac{x+y}{2} \geq \sqrt{xy}$

so $\frac{(x+y)^2}{4} \geq xy$

so $x^2 + 2xy + y^2 \geq 4xy$ so

so $x^2 - 2xy + y^2 \geq 0$

so $(x-y)^2 \geq 0$

I assume here that each of these steps is simple enough not to need a comment to justify it. But <u>the problem</u> is that the "argument" moves in the <u>wrong direction</u>. It begins with the statement that you want to prove, and ends with $(x-y)^2 \geq 0$. The words "so" indicate that the next line is true <u>because</u> the preceding line is true. This is really a proof that "If $\frac{x+y}{2} \geq \sqrt{xy}$, then $(x-y)^2 \geq 0$." (*which would be an odd thing to try to prove* since $(x-y)^2 \geq 0$ *is already known)*

If you remove the word "so"

$\frac{x+y}{2} \geq \sqrt{xy}$

$\frac{(x+y)^2}{4} \geq xy$

$x^2 + 2xy + y^2 \geq 4xy$

$x^2 - 2xy + y^2 \geq 0$

$(x-y)^2 \geq 0$

the argument <u>still</u> moves in the wrong direction.

One way to <u>fix</u> this (in this example) is in Column 2.

A PROOF

Because $x \geq 0$ and $y \geq 0$,
$\sqrt{(x+y)^2} = |x+y| = x+y$,
and $\sqrt{xy}$ is defined, we know that

$\frac{x+y}{2} \geq \sqrt{xy}$

$\Leftrightarrow \frac{(x+y)^2}{4} \geq xy$ (*)

$\Leftrightarrow x^2 + 2xy + y^2 \geq 4xy$

$\Leftrightarrow x^2 - 2xy + y^2 \geq 0$

$\Leftrightarrow (x-y)^2 \geq 0$ •

This proof does start with the line $\frac{x+y}{2} \geq \sqrt{xy}$ but each line is stated to be equivalent to the next line so that, in the end, $\frac{x+y}{2} \geq \sqrt{xy}$ is seen to be equivalent to the (true) statement $(x-y)^2 \geq 0$.

Of course, if you write " $\Leftrightarrow$ " between two lines, that equivalence might need some justification. The comments (*) at the beginning are meant to justify the implication $\Uparrow$ between from line 2 to line 1. The other equivalences here are simple enough not to need additional comment.

ANOTHER PROOF

We know that $(x-y)^2 \geq 0$, that is $x^2 - 2xy + y^2 \geq 0$. Therefore $x^2 + 2xy + y^2 \geq 4xy$, so $\frac{(x+y)^2}{4} \geq xy$.

Since $x \geq 0$ and $y \geq 0$, we have that $\sqrt{(x+y)^2} = |x+y| = x+y$, and also $\sqrt{xy}$ is defined. Therefore $\frac{x+y}{2} \geq \sqrt{xy}$. •

Here, the idea for the proof clearly came from looking at the incorrect "proof" given above. It was used a scratchwork. It begins with something known and ends with our conclusion.

"<u>Working backwards</u>" is <u>not</u> a proof (unless the steps can all be joined by $\Leftrightarrow$'s); working backwards is a way to get ideas for a proof which is then written down in the correct "direction."