Peano Systems and the Whole Number System
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We have a good informal picture about how the system of whole numbers works. By the whole number <u>system</u> we mean to <u>the set</u> $\omega = \{0, 1, 2, ... \}$, <u>together with</u> its rules for arithmetic and for handling inequalities (for example, if $a, b, c \in \omega$ and $a < b$, then $a + c < b + c$). Informally, we know a multitude of facts about behavior involving whole numbers. $+$, $\cdot$, $=$, $<$, and $\leq$. We also know how induction works.

Ultimately, we want to show how the whole number system can be described in terms of our foundation, set theory. We want to construct a system consisting of sets, ways to combine them ($+$, $\cdot$) and ways to compare them ($<$, $\leq$) so that the system "acts just like" the whole number system. As we have said several times, mathematicians don't care about what the whole numbers "really are." If we can use set theory to build a system that "acts just like ω", then all mathematicians can agree to treat that system as ω.

More carefully, what do we need to do? When have we got a system "that acts just like ω"? There are so many facts we know about the whole number system that we should build into this system of sets. There may even be about facts about ω that we don't know but that ought to be included. Our job seems like a hopeless task.

To make things more manageable, it would be very helpful if we had a short list of "the crucial properties" of ω — a list <u>from which</u> we can prove that the other important properties of ω must also inevitably be true. Then, if we can build a system of sets which has all "the crucial properties" of ω, then our new system will include the other important properties of ω automatically.

Fortunately, there is just such a short list — axioms developed by the mathematician Giuseppe Peano in 1889. The latter part of the 19th century, and the beginning of the20[th], were an "age of rigor" for mathematics — a period when firm foundations formathematics were being established. By then, for example, calculus had been around for a couple of centuries and seemed to work well — at least in skilled and sensitive hands — but it was clear that there was a lot of vagueness about why it all worked. Part of the problem had to do with not having firm foundations for the number systems (particularly $\mathbb{R}$).

We are going to look at the list of "Peano's Axioms" and try to indicate how all the informal properties of the whole number system ω follow from the properties in the list. There are many, many details to check. <u>We will check some of the details to indicate how (with several additional lectures) all the details could be ironed out. In not doing everything, there is no attempt to "hide" something hard. Any material we leave out is truly just "more of the same."</u>

Definition A <u>Peano system</u> $\mathcal{P}$ is a collection of objects with the following properties:

> P1) There is a special object in $\mathcal{P}$ named "0."
> *(Although the name "0" is intended to <u>suggest</u> "the whole number zero," we really know <u>nothing</u> about how the object called "0" in a Peano system acts except for what is stated in (or deducible from) the remaining axioms.*
>
> P2) For each object $x \in \mathcal{P}$, there is exactly one object in $\mathcal{P}$ called the <u>successor of x</u> (for short, we write x^+ to represent the successor of x).
>
> P3) 0 is not the successor of any object in $\mathcal{P}$:
>
> $$(\forall x \in \mathcal{P}) \ \ x^+ \neq 0$$
>
> P4) Different objects in $\mathcal{P}$ have different successors :
>
> $$(\forall x \in \mathcal{P})(\forall y \in \mathcal{P}) \ \ (x \neq y \Rightarrow x^+ \neq y^+)$$
>
> P5) Suppose $A \subseteq \mathcal{P}$. If $0 \in A$ and if $(\forall x \in \mathcal{P})\,(x \in A \Rightarrow x^+ \in A)$ is true, then $A = \mathcal{P}$.

Note: In his 1889 book, Peano went so far as to also include a few other axioms about how " $=$ " behaves: for example,

> $(\forall x \in \mathcal{P}) \ \ x = x$ *and*
> $(\forall x \in \mathcal{P})(\forall y \in \mathcal{P}) \ ((x = y) \ \Rightarrow y = x))$

Our point of view is that " $=$ " is a logical term meaning "is the same thing as" and that such assumptions about " $=$ " do not really need to be spelled out $-$ although doing so would certainly be harmless.

A Peano system is an "abstract system": we are given no information whatsoever about what the "objects" in $\mathcal{P}$ "really are," and we have no information about how x^+ can be found for a given $x \in \mathcal{P}$. <u>The only things we know about the objects in $\mathcal{P}$ and their successors is what the axioms P1-P5 say about their behavior</u>. Of course, we can logically deduce (prove) new pieces of information about $\mathcal{P}$ (theorems) from those axioms.

Until a reasonable collection of theorems about a Peano system is built up to use, the proofs of theorems will usually rely on axiom P5 $-$ which we will refer to as the induction axiom.

The challenge (and the amusement) of proving things about a Peano system is that we have so little, at the beginning, to work with. We have to fight for each little new fact. But the more things we prove, the more tools we have to work with and the easier it gets.

Notice that the informal whole number system, ω, obeys each of the axioms P1-P5 provided that

        i) <u>we interpret</u> the objects x in $\mathcal{P}$ to be whole numbers, and
        ii) <u>we interpret</u> "successor" x^+ to mean the whole number "$x + 1$."

Under this interpretation, ω is an example of a Peano system. Of course, axiom P5 is what we called the Principle of Mathematical Induction (PMI) in ω.

When we have an abstract system like $\mathcal{P}$ and we

        i) <u>interpret</u> all the objects and operations in $\mathcal{P}$ (such as "successor")
        as representing certain concrete objects and operations, and

        ii) all the assumptions about the objects/operations in the abstract system
        become true statements about the specific objects in the interpretation

then we say we have found a concrete <u>model</u> for the abstract system. Thus, <u>ω is a model for the abstract Peano system $\mathcal{P}$</u>.

Some Theorems About a Peano System $\mathcal{P}$

To illustrate dealing with an abstract system, we will prove some simple theorems about $\mathcal{P}$ that follow from P1-P5. (*Since the theorems follow logically from the axioms P1-P5, and because P1-P5 (as interpreted) are true statements in a model, each theorem must also be true (as interpreted) in any model of $\mathcal{P}$ (for example, in ω). For example, see the italicized interpretation of Theorem 1 in the model ω.*)

Theorem 1 For all $x \in \mathcal{P}$, either $x = 0$ or $(\exists y \in \mathcal{P})\ x = y^+$ (that is, every nonzero x in $\mathcal{P}$ is a successor). (*Interpreted in the model ω, Theorem 1 says that for each nonzero whole number x, there is a whole number y such that $x = y + 1$.*)

Proof Let $A = \{x \in \mathcal{P} : x = 0 \text{ or } (\exists y \in \mathcal{P})\ x = y^+\} = \{x \in \mathcal{P} : x = 0 \text{ or } x \text{ is a successor}\}$. We need to show (using P5) that $A = \mathcal{P}$.

    i) By definition of A, $0 \in A$
    ii) Suppose $x \in A$. Then $x^+ \in A$ because x^+ <u>is</u> a successor (namely, the successor
        of x).

By the induction axiom P5, we conclude that $A = \mathcal{P}$. $\bullet$

A <u>corollary</u> is a theorem that follows as a relatively quick and easy consequence of a previous theorem.

Corollary 2 If $x \in \mathcal{P}$ and $x \neq 0$, then $(\exists! \, y \in \mathcal{P}) \;\; x = y^+$.

Proof Theorem 1 gives that if $x \neq 0$, then $(\exists \, y \in \mathcal{P}) \;\; x = y^+$
     To show uniqueness, notice that if $x = y^+$ <u>and</u> $x = z^+$, then $y^+ = z^+$, so $y = z$
(using the contrapositive of P4).   ●

Definition   If $x = y^+$ in $\mathcal{P}$, we call y <u>the predecessor of x</u>.

Notice that the definition makes sense: we can say <u>the</u> predecessor because (from
Corollary 2) there can't be more than one predecessor for x. Corollary 2 therefore says
that each nonzero element x in $\mathcal{P}$ has a unique predecessor.

Theorem 3 For all $x \in \mathcal{P}$, $x \neq x^+$ (*that is, no object x in $\mathcal{P}$ is its own successor*).

Proof <u>Homework Exercise</u>

Theorem 4 *If $x \in \mathcal{P}$, then either $x = 0$ or x can be obtained from 0 by applying the*
successor operation to 0 a finite number of times.

Proof *Let $A = \{ x \in \mathcal{P} : \; x = 0$ or x can be obtained from 0 by applying the successor*
operation to 0 a finite number of times$\}$.

     $0 \in A$ (by definition of A)

     Suppose $x \in A$. We prove that $x^+ \in A$.

          If $x = 0$, then $x^+ \in A$ because $x^+ = 0^+$ can be obtained by applying the
          successor operation just <u>one</u> time.

          If $x \neq 0$, then (because $x \in A$) x can be obtained from 0 by a finite
          number of successor operations. But then one additional application of
          the successor operation produces x^+. Therefore $x^+ \in A$.

By the Induction Axiom P5), $A = \mathcal{P}$, which proves the theorem. ●

Corollary 5 *If $x, y \in \mathcal{P}$ and $x \neq y$, then one of x or y can be obtained from the other by*
applying the successor operation a finite number of times.

Proof *If one of x or y is 0 (say, $x = 0$) then Theorem 4 says we can obtain y by applying the successor operation to x a finite number of times.*

If neither x nor y is 0, then (by Theorem 4) we can obtain both x and y by from 0 using the successor operation. Applying the successor operation to 0, we arrive first at (say) x; and then continuing to apply the successor operation an additional number of times produces y.   •

<u>NOTE</u>:   Theorem 4 and Corollary 5 are italicized because we will not use them in any <u>proofs</u> that come later. In fact a "purist" might object that if we are trying to formally develop a theory of Peano systems <u>in order to define</u> the system of whole numbers ω, then we should not be allowed to use an argument that involves doing something "a <u>finite</u> number of times" − objecting that we can't formally say what "a finite number of times" <u>means</u> <u>until after</u> we have defined the whole number system.
Nevertheless, it seemed like it would be helpful to include the italicized results to help build up our intuitive picture of what a Peano system $\mathcal{P}$ "looks like" − as discussed in the next section.
*See Theorem 13.*

All Peano systems are "the same"

What does a Peano system "look like" ?   We can get an idea with a schematic diagram in which an arrow " → " points to "the successor."   We start with 0, which has no predecessor:

$$0 \ \rightarrow 0^+ \ \rightarrow (0^+)^+ \ \rightarrow \ ... \quad \rightarrow x \ \rightarrow \ x^+ \ \rightarrow \ ...$$

Theorem 4 tells us that every nonzero object $x \in \mathcal{P}$ appears in this diagram eventually, after applying the successor a sufficient number of times.

When we make the diagram, it will always "keep on going forward" – that is, there will never be any "backward loops" like
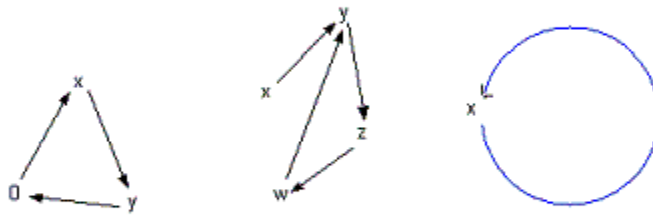
Which axiom says that the first loop is impossible? the second? Why is the third loop impossible ?

Thus we can informally picture a Peano system $\mathcal{P}$ as an "infinite linear chain" starting at its special element, 0:

$$0 \ \rightarrow 0^+ \ \rightarrow (0^+)^+ \ \rightarrow \ ... \quad \rightarrow x \ \rightarrow \ x^+ \ \rightarrow \ ... \text{ (and so on, forever)}$$

All Peano systems must look the same. The technical phrase for this is that <u>all Peano Systems are isomorphic</u>. To be a little more precise, this means that if we have two Peano systems $\mathcal{P}$ and $\boldsymbol{\mathcal{P}}$ (we use **boldface** for the second Peano system and its objects), then it is possible

       i) to pair off all the elements of $\mathcal{P}$ with all the elements of $\boldsymbol{\mathcal{P}}$ in such a way that so that each object in one system has a unique "partner" in the other system.

       ii) to do this not just with some "random" pairing, but to do it in such a way that 0 is paired with $\boldsymbol{0}$ and the pairing respects the successor operation: if $x \in \mathcal{P}$ is partnered with $\boldsymbol{x} \in \boldsymbol{\mathcal{P}}$, then x^+ (in $\mathcal{P}$) is partnered with $\boldsymbol{x}^+$ (in $\boldsymbol{\mathcal{P}}$)) : in other words, "the successor of partner is the partner of the successor."

Our images of these systems would then look like this – where vertical arrows indicate the "pairing":

$$0 \;\rightarrow\; 0^+ \;\rightarrow\; (0^+)^+ \;\rightarrow ... \;\rightarrow\; \quad x \;\rightarrow\; x^+ \;\rightarrow ... \qquad \text{(and so on, forever)}$$
$$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad\qquad\qquad \updownarrow \qquad \updownarrow$$
$$\boldsymbol{0} \;\rightarrow\; \boldsymbol{0}^+ \;\rightarrow\; \boldsymbol{(0}^+\boldsymbol{)}^+ \;\rightarrow ... \;\rightarrow\; \quad \boldsymbol{x} \;\rightarrow\; \boldsymbol{x}^+ \;\rightarrow ... \qquad \text{(and so on, forever)}$$

A slightly different way to think of this isomorphic "pairing" is just to imagine that each object x in $\mathcal{P}$ has been "renamed" subject to the following rules:

       i) 0 is renamed as $\boldsymbol{0}$
       ii) if x is renamed as $\boldsymbol{x}$, then x^+ is renamed as $\boldsymbol{x}^+$

From this point of view, the "second" Peano System $\boldsymbol{\mathcal{P}}$ is just the "same old stuff" but with new names.

This is an example of an important phenomenon. Sometimes different systems really are complete look-alikes: one is just the other with elements "renamed" in a way that respects the operations inside the system (e.g., "successor"). The systems are perfect "mirror images" of each other – <u>they have exactly the same structure</u>. The words "structure" and "system" are a little vague, so we can't make a precise mathematical definition here. But here is an informal definition that may be useful to remember.

Informal Definition   Suppose there is a "pairing (or renaming) rule" between two systems which pairs off all the objects in the two systems with each other in a one-for-one way. Suppose, moreover, that this pairing is done in a way that respects all the operations (like "successor", for example) in the systems. Then we say that the two structures are <u>isomorphic</u> and the "pairing rule" is called an <u>isomorphism between the structures</u>.

Note: "isomorphism" comes from two Greek words,

> | *"isos"* | *meaning* | *"equal" or "same"* |
> | --- | --- | --- |
> | *"morphe"* | *meaning* | *"shape" or "form" or "structure")* |

To make the definition more precise, we would replace "pairing rule" with "a one-to-one, onto function" between the systems. But that additional precision needs to wait until we say more about functions, one-to-one functions, onto functions, etc.

Students who have already taken Math 309 (Matrix Algebra) should have seen the idea of "isomorphic systems" before – although the word "isomorphic" might not have been used. If V is a finite dimensional vector space with basis $\mathcal{B} = \{b_1, ..., b_n\}$, then V is isomorphic to ("looks just like") the vector space $\mathbb{R}^n$. The "coordinate mapping" pairs off each vector $x \in V$ with a vector in $\mathbb{R}^n$, namely, $x \rightleftarrows (c_1, ..., c_n)$ where $c_1, ..., c_n$ are the coordinates of x with respect to the basis $\mathcal{B}$.)

This is sufficient detail for what we are going to do. We have argued that any two Peano systems are isomorphic, so that "if you've seen one Peano system, you've seen them all."

> However, those who are interested are encouraged to also read this optional (indented) material. Unlike the more informal discussion, above, the following discussion makes no use of the "picture" and makes no use of the italicized results *Theorem 4* and *Corollary 5*. The "renaming" or "pairing" rule is defined inductively without any reference to the figures above.
>
> Define a "renaming" rule (function) $\boldsymbol{R}$ that pairs each element in $\mathcal{P}$ with a "unique partner" in the other Peano system $\boldsymbol{\mathcal{P}}$. The definition of $\boldsymbol{R}$ is done <u>inductively</u> (that is, using axiom P5):
>
> > Let $\boldsymbol{R}(0) = \boldsymbol{0}$
> > and, $\forall x \in \mathcal{P} \quad \boldsymbol{R}(x^+) = (\boldsymbol{R}(x))^+ \quad (*)$
>
> $(*)$ tells you how to find $\boldsymbol{R}(x^+)$ (an object in $\boldsymbol{\mathcal{P}}$) if you already know $\boldsymbol{R}(x)$ (an object in $\boldsymbol{\mathcal{P}}$). This defines $\boldsymbol{R}$ for every $x \in \mathcal{P}$:
>
> > For example, the rule gives $\quad \boldsymbol{R}(0) = \boldsymbol{0}$,
> > $\boldsymbol{R}(0^+) = (\boldsymbol{R}(0))^+ = \boldsymbol{0}^+$,
> > $\boldsymbol{R}((0^+)^+) = (\boldsymbol{R}(0^+))^+ = (\boldsymbol{0}^+)^+$, etc.
>
> > More precisely, if we let $A = \{x \in \mathcal{P} : \boldsymbol{R}(x)$ is defined$\}$, then $0 \in A$ and if $x \in A$, then $x^+ \in A$ – so, by P5), $A = \mathcal{P}$.

> There are two important observations to make:

1) <u>Different</u> elements $x, y \in \mathcal{P}$ get assigned to <u>different</u> partners in
$\mathcal{P}$ – that is, if $x \neq y$, then $\boldsymbol{R}(x) \neq \boldsymbol{R}(y)$. To see this, we use induction.

Let $A = \{x \in \mathcal{P} : \forall y \ (y \neq x \Rightarrow \boldsymbol{R}(y) \neq \boldsymbol{R}(x) \ )\}$. We want to
see that $A = \mathcal{P}$.

$0 \in A$: To see this, we need to check that if $y \neq 0$, then
$\boldsymbol{R}(y) \neq \boldsymbol{R}(0) = \boldsymbol{0}$. In other words, we have to check
that a nonzero y in $\mathcal{P}$ gets a nonzero partner in $\mathcal{P}$.

Since $y \neq 0$, then (by Corollary 2) $y = z^+$ for
some $z \in \mathcal{P}$. Therefore $\boldsymbol{R}(y) = \boldsymbol{R}(z^+)$
$= (\boldsymbol{R}(z))^+$. That means that $\boldsymbol{R}(y)$ has a
predecessor $\boldsymbol{R}(z)$ in $\mathcal{P}$. But $\boldsymbol{0}$ has no predecessor
in $\mathcal{P}$ (by P3), so $\boldsymbol{R}(y) \neq \boldsymbol{0}$.

If $x \in A$, we must show that $x^+ \in A$, that is: we must
show that if $y \neq x^+$, then $\boldsymbol{R}(y) \neq \boldsymbol{R}(x^+)$. We do this by
showing the contrapositive: if $\boldsymbol{R}(y) = \boldsymbol{R}(x^+)$, then
$y = x^+$.

Suppose $\boldsymbol{R}(y) = \boldsymbol{R}(x^+)$. Since $x^+ \neq 0$ (by P3),
$\boldsymbol{R}(x^+) \neq \boldsymbol{R}(0)$ (since $0 \in A$) so $y \neq 0$. Therefore
y has a predecessor, say $y = z^+$.

Then $(\boldsymbol{R}(x))^+ = \boldsymbol{R}(x^+) = \boldsymbol{R}(z^+) = (\boldsymbol{R}(z))^+$.
By P4), we conclude that $\boldsymbol{R}(x) = \boldsymbol{R}(z)$. Since
$x \in A$, this means that $x = z$. But then
$y = z^+ = x^+$.

Therefore, by P5), $A = \mathcal{P}$. ●

2) <u>Every object in $\mathcal{P}$ acquires a partner from $\mathcal{P}$</u>. Again, we use induction.
Let $\boldsymbol{A} = \{\boldsymbol{x} \in \mathcal{P} : \boldsymbol{x} = \boldsymbol{R}(y) \text{ for some } y \in \mathcal{P}\}$. We need to show that
$\boldsymbol{A} = \mathcal{P}$.

$\boldsymbol{0} \in \boldsymbol{A}$ because $\boldsymbol{0} = \boldsymbol{R}(0)$

Suppose $\boldsymbol{x} \in \boldsymbol{A}$. Then $\boldsymbol{x} = \boldsymbol{R}(y)$ for some $y \in \mathcal{P}$.
Therefore $\boldsymbol{R}(y^+) = (\boldsymbol{R}(y))^+ = \boldsymbol{x}^+$. In other words, $\boldsymbol{x}^+$
is partnered with y^+ from $\mathcal{P}$, so $\boldsymbol{x}^+ \in \boldsymbol{A}$.

By P5), $\boldsymbol{A} = \mathcal{P}$.

Putting observations 1) and 2) together, the rule R gives an exact pairing, one-for-one (R is a "one-to-one, onto function") between the all the objects in $\mathcal{P}$ and all those in $\mathcal{P}$. By the definition of R, the pairing respects the successor operation work in the two systems:

$$R(x^+) = R(x)^+$$

"the partner of the successor" $=$ "the successor of the partner"

More about a Peano System

We want to convince ourselves that a Peano system captures the essence of our informal system ω. Already, we have a "mental picture" and a few theorems which suggest that the objects in a Peano system are arranged just like the whole numbers. We want to see that we can define "addition," "multiplication," and " $<$ " between objects in a Peano system and that, when we're done, the result acts just like ω.

All Peano systems look alike, so let's begin by assigning some convenient <u>names</u> to the objects in a Peano system. After all, needing to write things like $0^{+++++++}$ becomes tedious.

There are <u>lots of possible ways to name things</u>.   For example, some possibilities could be:

	0	0^+	0^{++}	0^{+++}	0^{++++}	0^{+++++}	0^{++++++}	... etc.
Naming System	œ	†	‡	¿	ð	€	ß	
Naming System	0	I	II	III	IV	V	VI	...
Naming System	0	1	10	11	100	101	110	...
Naming System	0	1	2	3	4	5	6	...

The point is that there are lots of ways to <u>invent names</u> for $0, 0^+, 0^{++}$, ...etc. It's important, here, to remember that however we decide to invent names for the objects in the Peano system, the names themselves don't give us any new <u>information</u>. But, keeping that in mind, we might as well use names are convenient and that remind us of how we <u>hope</u> the system is going to work. So we'll use the intuitively familiar symbols $0, 1, 2, 3, ...$ as in the fourth row of the table.

> <u>Caution</u> For now, $0, 1, 2...$ are now just "marks on paper" $-$ the <u>names</u>
> we're giving objects in the Peano system. There's no more reason to say "1 plus 2
> $= 3$" than there is to say "† plus ‡ $= $ ¿" : both are just ways of saying (in
> different naming systems) that "0^+ plus $0^{++} = 0^{+++}$" $-$ and in fact, the
> statement "0^+ plus $0^{++} = 0^{+++}$" <u>has no meaning yet at all</u> because we haven't
> <u>defined</u> what "<u>plus</u>" means in a Peano system.
>
> We <u>cannot</u> say $2 \cdot 2 = 4$, because (right now) that statement is just a new
> way of writing $0^{++} \cdot 0^{++} = 0^{++++}$ $-$ <u>which, at the moment, has no meaning at
> all</u>, because we haven't even defined what it means to "multiply" objects in a

Peano System.

We can, however, use these new names now to record things that we <u>do</u> already know. For example, the axioms for a Peano system now read:

P1) There is one special object named "0" in $\mathcal{P}$

P2) For each object $n \in \mathcal{P}$, there is exactly one object in $\mathcal{P}$ called its <u>successor</u> (and denoted n^+)

P3) 0 is not the successor of any object, that is, $(\forall n \in \mathcal{P})\ \ n^+ \neq 0$

P4) Different objects have different successors, that is
$$\forall m, \forall n \in \mathcal{P}\ (m \neq n \Rightarrow m^+ \neq n^+)$$

P5) Suppose $A \subseteq \mathcal{P}$. If $0 \in A$ and if
$$(\forall n \in \mathcal{P})\ \ (n \in A \Rightarrow n^+ \in A)$$
then $A = \mathcal{P}$.

<u>Just because of how we named things</u>, statements like these are true:

$$0^+ = 1\ \ \text{(i.e., 1 is the successor of 0)},$$
$$0^{++} = 1^+ = 2,$$
$$4^+ = 5$$

If we had decided instead to use the naming system in the first row of the table, the following would be true:

$$\text{œ}^+ = \dagger$$
$$\text{œ}^{++} = \dagger^+ = \ddagger$$
$$\eth^+ = \text{€}$$

The theorems we already proved, with the new naming system, can now be written:

Theorem 1 For all $n \in \mathcal{P}$, either $n = 0$ or $n = m^+$ for some $m \in \mathcal{P}$

Corollary 2 If $n \in \mathcal{P}$ and $n \neq 0$, then $n = m^+$ for a <u>unique</u> $m \in \mathcal{P}$.

Theorem 3 For all $n \in \mathcal{P}$, $n \neq n^+$

Theorem 4 *If $n \in \mathcal{P}$, then $n = 0$ or n can be obtained from 0 by applying the successor operation finitely often.*

Corollary 5 *If $m, n \in \mathcal{P}$ and $m \neq n$, then one of m and n can be obtained from the other by applying the successor operation finitely often.*