Peano Systems and the Whole Number System

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We have a good informal picture about how the whole numbers work. By the whole number system we mean to the set $\omega = \{0, 1, 2, ...\}$, together with its rules for arithmetic and for handling inequalities (for example, if $a, b, c \in \omega$ and $a < b$, then $a + c < b + c$). Informally, we know a multitude of facts about behavior involving whole numbers. $+$, $\cdot$, $=$, $<$, and $\leq$. We also know how induction works.

Ultimately, we want to show how the whole number system can be described in terms of our foundation, set theory. We want to construct a system consisting of sets, ways to combine them ($+$, $\cdot$) and ways to compare them ($<$, $\leq$) so that the system "acts just like" the whole number system. As we have said several times, mathematicians don't care about what the whole numbers "really are." If we can use set theory to build a system that "acts just like ω", then all mathematicians can agree to call that system ω .

More carefully, what do we need to do? When have we got a system "that acts just like ω"? There are so many facts we know about the whole number system that we should build into this system of sets. There may even be about facts about ω that we don't know but that ought to be included. Our job seems like a hopeless task.

To make things more manageable, it would be very helpful if we had a short list of "the crucial properties" of ω – a list from which we can prove that the other important properties of ω must also inevitably be true. Then, if we can build a system of sets which has all "the crucial properties" of ω, then our new system will include the other important properties of ω automatically.

Fortunately, there is just such a short list – axioms developed by the mathematician Giuseppe Peano in 1889. The latter part of the 19th century, and the beginning of the20th, were an "age of rigor" for mathematics – a period when firm foundations for mathematics were being established. This project was felt to be intellectually necessary. For example, calculus had by then been around for a couple of centuries and seemed to work well – at least in skilled and sensitive hands. But there was clearly a lot of vagueness about why it worked. A lack of firm foundations for the number systems (partilcularly $\mathbb{R}$) was part of the problem.

We are going to look at the list of "Peano's Axioms" and try to indicate how all the informal properties of the whole number system ω follow from the properties in the list. There are many, many details to check. We will check some of the details to indicate how (with several additional lectures) all the details could be ironed out. In not doing everything, there is no attempt to "hide" something hard. Any material we leave out is truly just "more of the same."

Definition A <u>Peano system</u> $\mathcal{P}$ is a collection of objects with the following properties:

P1) There is a special object in $\mathcal{P}$ named "0."
$($*Although the name "* 0*" is intended to <u>suggest</u> "the whole number zero," we really know <u>nothing</u> about how the object called "* 0*" in a Peano system acts except for what is stated in (or deducible from) the remaining axioms.*

P2) For each object $x \in \mathcal{P}$, there is exactly one object in $\mathcal{P}$ called the <u>successor of x</u> (for short, we write x^+ to represent the successor of x).

P3) 0 is not the successor of any object in $\mathcal{P}$:

$$(\forall x \in \mathcal{P}) \ \ x^+ \neq 0$$

P4) Different objects in $\mathcal{P}$ have different successors :

$$(\forall x \in \mathcal{P})(\forall y \in \mathcal{P}) \ \ (x \neq y \Rightarrow x^+ \neq y^+)$$

P5) Suppose $A \subseteq \mathcal{P}$. If $0 \in A$ and if $(\forall x \in \mathcal{P})\,(x \in A \Rightarrow x^+ \in A)$ is true, then $A = \mathcal{P}$.

Note: In his 1889 book, Peano went so far as to also include a few other axioms about how " $=$ *" behaves: for example,*

$$(\forall x \in \mathcal{P}) \ \ x = x \ \ \textit{and}$$
$$(\forall x \in \mathcal{P})(\forall y \in \mathcal{P}) \ ((x = y) \ \Rightarrow (y = x))$$

Our point of view is that " $=$ *" is a logical term meaning "is the same thing as" and that such assumptions about "* $=$ *" do not really need to be spelled out* $-$ *although doing so would certainly be harmless.*

A Peano system is an "abstract system" : we are given no information whatsoever about what the "objects" in $\mathcal{P}$ "really are," and we have no information about how x^+ can be found for a given $x \in \mathcal{P}$. <u>The only things we know about the objects in $\mathcal{P}$ and their successors is what the axioms P1-P5 say about their behavior</u>. Of course, we can logically deduce (prove) new pieces of information about $\mathcal{P}$ (theorems) from those axioms.

Until a reasonable collection of theorems about a Peano system is built up to use, the proofs of theorems will usually rely on axiom P5 $-$ which we will refer to as the induction axiom in $\mathcal{P}$.

The challenge (and the amusement) of proving things about a Peano system is that we have so little, at the beginning, to work with. We have to fight for each little new fact. But the more things we prove, the more tools we have to work with and the easier it gets.

Notice that the informal whole number system, ω, obeys each of the axioms P1-P5 provided that

i) we interpret the objects x in $\mathcal{P}$ to be whole numbers, and
ii) we interpret "successor" x^+ to mean the whole number "$x+1$."

Under this interpretation, ω is an example of a Peano system. Of course, axiom P5 is what we called the Principle of Mathematical Induction (PMI) in ω.

When we have an abstract system like $\mathcal{P}$ and we

i) interpret all the objects and operations in $\mathcal{P}$ (such as "successor") as representing certain concrete objects and operations, and

ii) all the assumptions about the objects/operations in the abstract system become true statements about the specific objects in the interpretation

then we say we have found a concrete model for the abstract system. Thus, ω is a model for the abstract Peano system $\mathcal{P}$.

Some Theorems About a Peano System $\mathcal{P}$

To illustrate dealing with an abstract system, we will prove some simple theorems about $\mathcal{P}$ that follow from P1-P5. (*The theorems follow logically from the axioms P1-P5. Because P1-P5 (as interpreted in the model ω), each theorem must also be true when interpreted the same way as as statement about ω. For example, see the italicized interpretation of Theorem 1 in the model ω.*)

Theorem 1 For all $x \in \mathcal{P}$, either $x = 0$ or $(\exists y \in \mathcal{P})\ x = y^+$ (that is, every nonzero x in $\mathcal{P}$ is a successor). (*Interpreted in the model ω, Theorem 1 says that for each nonzero whole number x, there is a whole number y such that $x = y + 1$.*)

Proof Let $A = \{x \in \mathcal{P} : x = 0$ or $(\exists y \in \mathcal{P})\ x = y^+\} = \{x \in \mathcal{P} : x = 0$ or x is a successor$\}$. We need to show (using P5) that $A = \mathcal{P}$.

i) By definition of A, $0 \in A$
ii) Suppose $x \in A$. Then $x^+ \in A$ because x^+ is a successor (namely, the successor of x).

By the induction axiom P5, we conclude that $A = \mathcal{P}$. $\bullet$

A <u>corollary</u> is a theorem that follows as a relatively quick and easy consequence of a previous theorem.

Corollary 2 If $x \in \mathcal{P}$ and $x \neq 0,$ then $(\exists!\, y \in \mathcal{P})\ x = y^+.$

Proof Theorem 1 gives that if $x \neq 0$, then $(\exists\, y \in \mathcal{P})\ x = y^+$
To show uniqueness, notice that if $x = y^+$ <u>and</u> $x = z^+,$ then $y^+ = z^+$, so $y = z$ (using the contrapositive of P4). •

Definition If $x = y^+$ in $\mathcal{P}$, we call y <u>the predecessor of x</u>.

Notice that the definition makes sense: we can say <u>the</u> predecessor because (from Corollary 2) there can't be more than one predecessor for x. Corollary 2 therefore says that each nonzero element x in $\mathcal{P}$ has a unique predecessor.

Theorem 3 For all $x \in \mathcal{P},\ x \neq x^+$ (*that is, no object x in $\mathcal{P}$ is its own successor*).

Proof <u>Homework exercise</u>

Theorem 4 *If $x \in \mathcal{P}$, then either $x = 0$ or x can be obtained from 0 by applying the successor operation to 0 a finite number of times.*

Proof *Let $A = \{x \in \mathcal{P} :\ x = 0$ or x can be obtained from 0 by applying the successor operation to 0 a finite number of times$\}$.*

$0 \in A$ *(by definition of A)*

Suppose $x \in A$. We prove that $x^+ \in A$.

If $x = 0,$ then $x^+ \in A$ because $x^+ = 0^+$ can be obtained by applying the successor operation just <u>one</u> time.

If $x \neq 0$, then (because $x \in A$) x can be obtained from 0 by a finite number of successor operations. But then one additional application of the successor operation produces x^+. Therefore $x^+ \in A$.

By the Induction Axiom P5), $A = \mathcal{P}$, which proves the theorem. •

Corollary 5 *If $x, y \in \mathcal{P}$ and $x \neq y$, then one of x or y can be obtained from the other by applying the successor operation a finite number of times.*

Proof *If one of x or y is 0 (say, $x = 0$) then Theorem 4 says we can obtain y by applying the successor operation to x a finite number of times.*

If neither x nor y is 0, then (by Theorem 4) we can obtain both x and y by from 0 using the successor operation. Applying the successor operation to 0, we arrive first at (say) x; and then continuing to apply the successor operation an additional number of times produces y. ●

NOTE: Theorem 4 and Corollary 5 are italicized because we will not use them in any proofs that come later. In fact a "purist" might object that if we are trying to formally develop a theory of Peano systems in order to define the system of whole numbers ω, then we should not be allowed to use an argument that involves doing something "a finite number of times" – objecting that we can't formally say what "a finite number of times" means until after we have defined the whole number system.

Nevertheless, it seemed like it would be helpful to include the italicized results to help build up our intuitive picture of what a Peano system $\mathcal{P}$ "looks like" – as discussed in the next section.

See Theorem 13.

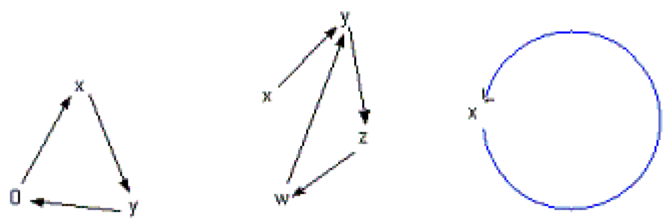
All Peano systems are "the same"

What does a Peano system "look like" ? We can get an idea with a schematic diagram in which an arrow " $\rightarrow$ " points to "the successor." We start with 0, which has no predecessor:

$$0 \rightarrow 0^+ \rightarrow (0^+)^+ \rightarrow \ldots \quad \rightarrow x \rightarrow x^+ \rightarrow \ldots$$

Theorem 4 tells us that every nonzero object $x \in \mathcal{P}$ appears in this diagram eventually, after applying the successor a sufficient number of times.

When we make the diagram, it will always "keep on going forward" – that is, there will never be any "backward loops" like



Which axiom says that the first loop is impossible? the second? Why is the third loop impossible ?

Thus we can informally picture a Peano system $\mathcal{P}$ as an "infinite linear chain" starting at its special element, 0:

$$0 \rightarrow 0^+ \rightarrow (0^+)^+ \rightarrow \ldots \quad \rightarrow x \rightarrow x^+ \rightarrow \ldots \text{ (and so on, forever)}$$

All Peano systems must look the same. The technical phrase for this is that <u>all Peano Systems are isomorphic</u>. To be a little more precise, this means that if we have two Peano systems $\mathcal{P}$ and $\boldsymbol{\mathcal{P}}$ (we use **boldface** for the second Peano system and its objects), then it is possible

i) to pair off all the elements of $\mathcal{P}$ with all the elements of $\boldsymbol{\mathcal{P}}$ in such a way that so that each object in one system has a unique "partner" in the other system.

ii) to do this not just with some "random" pairing, but to do it in such a way that 0 is paired with $\mathbf{0}$ and the pairing respects the successor operation: if $x \in \mathcal{P}$ is partnered with $\boldsymbol{x} \in \boldsymbol{\mathcal{P}}$, then x^+ (in $\mathcal{P}$) is partnered with $\boldsymbol{x^+}$ (in $\boldsymbol{\mathcal{P}}$)) : in other words, "the successor of partner is the partner of the successor."

Our images of these systems would then look like this $-$ where vertical arrows indicate the "pairing":

$$\begin{array}{ccccccccccccl} 0 & \rightarrow & 0^+ & \rightarrow & (0^+)^+ & \rightarrow & \ldots & \rightarrow & x & \rightarrow & x^+ & \rightarrow \ldots & \text{(and so on, forever)} \\ \updownarrow & & \updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow & & \\ \mathbf{0} & \rightarrow & \mathbf{0^+} & \rightarrow & \mathbf{(0^+)^+} & \rightarrow & \ldots & \rightarrow & \boldsymbol{x} & \rightarrow & \boldsymbol{x^+} & \rightarrow \ldots & \text{(and so on, forever)} \end{array}$$

A slightly different way to think of this isomorphic "pairing" is just to imagine that each object x in $\mathcal{P}$ has been "renamed" subject to the following rules:

i) 0 is renamed as $\mathbf{0}$
ii) if x is renamed as $\boldsymbol{x}$, then x^+ is renamed as $\boldsymbol{x^+}$

From this point of view, the "second" Peano System $\boldsymbol{\mathcal{P}}$ is just the "same old stuff" but with new names.

This is an example of an important phenomenon. Sometimes different systems really are complete look-alikes: one is just the other with elements "renamed" in a way that respects the operations inside the system (e.g., "successor"). The systems are perfect "mirror images" of each other $-$ <u>they have exactly the same structure</u>. The words "structure" and "system" are a little vague, so we can't make a precise mathematical definition here. But here is an informal definition that may be useful to remember.

Informal Definition Suppose there is a "pairing (or renaming) rule" between two systems which pairs off all the objects in the two systems with each other in a one-for-one way. Suppose, moreover, that this pairing is done in a way that respects all the operations (like "successor", for example) in the systems. Then we say that the two structures are <u>isomorphic</u> and the "pairing rule" is called an <u>isomorphism between the structures</u>.

Note: "isomorphism" comes from two Greek words,

"isos"	*meaning*	*"equal" or "same"*
"morphe"	*meaning*	*"shape" or "form" or "structure")*

To make the definition more precise, we would replace "pairing rule" with "a one-to-one, onto function" between the systems. But that additional precision needs to wait until we say more about functions, one-to-one functions, onto functions, etc.

Students who have already taken Math 309 (Matrix Algebra) should have seen the idea of "isomorphic systems" before – although the word "isomorphic" might not have been used. If V is a finite dimensional vector space with basis $\mathcal{B} = \{b_1, ..., b_n\}$, then V is isomorphic to ("looks just like") the vector space $\mathbb{R}^n$. The "coordinate mapping" pairs off each vector $x \in V$ with a vector in $\mathbb{R}^n$, namely, $x \rightleftarrows (c_1, ..., c_n)$ where $c_1, ..., c_n$ are the coordinates of x with respect to the basis $\mathcal{B}$.)

This is sufficient detail for what we are going to do. We have argued that any two Peano systems are isomorphic, so that "<u>if you've seen one Peano system, you've seen them all</u>."

> However, those who are interested are encouraged to also read this optional (indented) material. Unlike the more informal discussion, above, the following discussion makes no use of the "picture" and makes no use of the italicized results *Theorem 4* and *Corollary 5*. The "renaming" or "pairing" rule is defined inductively without any reference to the figures above.
>
> Define a "renaming" rule (function) $\boldsymbol{R}$ that pairs each element in $\mathcal{P}$ with a "unique partner" in the other Peano system $\boldsymbol{\mathcal{P}}$. The definition of $\boldsymbol{R}$ is done <u>inductively</u> (that is, using axiom P5):
>
> Let $\boldsymbol{R}(0) = \mathbf{0}$
> and, $\forall x \in \mathcal{P} \quad \boldsymbol{R}(x^+) = (\boldsymbol{R}(x))^{\boldsymbol{+}} \quad (*)$
>
> $(*)$ tells you how to find $\boldsymbol{R}(x^+)$ (an object in $\boldsymbol{\mathcal{P}}$) if you already know $\boldsymbol{R}(x)$ (an object in $\boldsymbol{\mathcal{P}}$). This defines $\boldsymbol{R}$ for every $x \in \mathcal{P}$:
>
> For example, the rule gives $\boldsymbol{R}(0) = \mathbf{0}$,
> $\boldsymbol{R}(0^+) = (\boldsymbol{R}(0))^{\boldsymbol{+}} = \mathbf{0}^{\boldsymbol{+}}$,
> $\boldsymbol{R}((0^+)^+) = (\boldsymbol{R}(0^+))^{\boldsymbol{+}} = (\mathbf{0}^{\boldsymbol{+}})^{\boldsymbol{+}}$, etc.
>
> More precisely, if we let $A = \{x \in \mathcal{P} : \boldsymbol{R}(x) \text{ is defined}\}$, then $0 \in A$ and if $x \in A$, then $x^+ \in A$ – so, by P5), $A = \mathcal{P}$.
>
> There are two important observations to make:

1) Different elements $x, y \in \mathcal{P}$ get assigned to different partners in $\boldsymbol{\mathcal{P}}$ – that is, if $x \neq y$, then $\boldsymbol{R}(x) \neq \boldsymbol{R}(y)$**.** To see this, we use induction.

Let $A = \{x \in \mathcal{P} : \forall y \ (y \neq x \ \Rightarrow \ \boldsymbol{R}(y) \neq \boldsymbol{R}(x) \ )\}$. We want to see that $A = \mathcal{P}$.

$0 \in A$: To see this, we need to check that if $y \neq 0$, then $\boldsymbol{R}(y) \neq \boldsymbol{R}(0) = \mathbf{0}$. In other words, we have to check that a nonzero y in $\mathcal{P}$ gets a nonzero partner in $\boldsymbol{\mathcal{P}}$.

Since $y \neq 0$, then (by Corollary 2) $y = z^+$ for some $z \in \mathcal{P}$. Therefore $\boldsymbol{R}(y) = \boldsymbol{R}(z^+)$ $= (\boldsymbol{R}(z))^+$. That means that $\boldsymbol{R}(y)$ has a predecessor $\boldsymbol{R}(z)$ in $\boldsymbol{\mathcal{P}}$. But $\mathbf{0}$ has no predecessor in $\boldsymbol{\mathcal{P}}$ (by P3), so $\boldsymbol{R}(y) \neq \mathbf{0}$**.**

If $x \in A$, we must show that $x^+ \in A$, that is: we must show that if $y \neq x^+$, then $\boldsymbol{R}(y) \neq \boldsymbol{R}(x^+)$. We do this by showing the contrapositive: if $\boldsymbol{R}(y) = \boldsymbol{R}(x^+)$, then $y = x^+$.

Suppose $\boldsymbol{R}(y) = \boldsymbol{R}(x^+)$. Since $x^+ \neq 0$ (by P3), $\boldsymbol{R}(x^+) \neq \boldsymbol{R}(0)$ (since $0 \in A$) so $y \neq 0$. Therefore y has a predecessor, say $y = z^+$.

Then $(\boldsymbol{R}(x))^+ = \boldsymbol{R}(x^+) = \boldsymbol{R}(z^+) = (\boldsymbol{R}(z))^+$**.** By P4), we conclude that $\boldsymbol{R}(x) = \boldsymbol{R}(z)$. Since $x \in A$, this means that $x = z$. But then $y = z^+ = x^+$.

Therefore, by P5), $A = \mathcal{P}$. •

2) Every object in $\boldsymbol{\mathcal{P}}$ acquires a partner from $\mathcal{P}$. Again, we use induction. Let $\boldsymbol{A} = \{\boldsymbol{x} \in \boldsymbol{\mathcal{P}} : \boldsymbol{x} = \boldsymbol{R}(y)$ for some $y \in \mathcal{P}\}$. We need to show that $\boldsymbol{A} = \boldsymbol{\mathcal{P}}$.

$\mathbf{0} \in \boldsymbol{A}$ because $\mathbf{0} = \boldsymbol{R}(0)$

Suppose $\boldsymbol{x} \in \boldsymbol{A}$**.** Then $\boldsymbol{x} = \boldsymbol{R}(y)$ for some $y \in \mathcal{P}$. Therefore $\boldsymbol{R}(y^+) = (\boldsymbol{R}(y))^+ = \boldsymbol{x}^+$. In other words, $\boldsymbol{x}^+$ is partnered with y^+ from $\mathcal{P}$, so $\boldsymbol{x}^+ \in \boldsymbol{A}$.

By P5), $\boldsymbol{A} = \boldsymbol{\mathcal{P}}$**.**

Putting observations 1) and 2) together, the rule $\boldsymbol{R}$ gives an exact pairing, one-for-one ($\boldsymbol{R}$ is a "one-to-one, onto function") between the all the objects in $\mathcal{P}$ and all those in $\boldsymbol{\mathcal{P}}$. By the definition of $\boldsymbol{R}$, the pairing respects the successor operation work in the two systems:

$$\boldsymbol{R}(x^+) = \boldsymbol{R}(x)^+$$

"the partner of the successor" $=$ "the successor of the partner"

More about a Peano System

We want to convince ourselves that a Peano system captures the essence of our informal system ω. Already, we have a "mental picture" and a few theorems which suggest that the objects in a Peano system are arranged just like the whole numbers. We want to see that we can define "addition," "multiplication," and " $<$ " between objects in a Peano system and that, when we're done, the result acts just like ω.

All Peano systems look alike, so let's begin by assigning some convenient names to the objects in a Peano system. After all, needing to write things like $0^{+++++++}$ becomes tedious.

There are lots of possible ways to name things. For example, some possibilities could be:

	0	0^+	0^{++}	0^{+++}	0^{++++}	0^{+++++}	0^{++++++}	... etc.
Naming System	œ	†	‡	¿	ð	€	ß	
Naming System	0	I	II	III	IV	V	VI	...
Naming System	0	1	10	11	100	101	110	...
Naming System	0	1	2	3	4	5	6	...

The point is that there are lots of ways to invent names for $0,\ 0^+, 0^{++},$ ...etc. It's important, here, to remember that however we decide to invent names for the objects in the Peano system, the names themselves don't give us any new information. But, keeping that in mind, we might as well use names are convenient and that remind us of how we hope the system is going to work. So we'll use the intuitively familiar symbols $0, 1, 2, 3, ...$ as in the fourth row of the table.

Caution For now, $0, 1, 2...$ are now just "marks on paper" $-$ the names we're giving objects in the Peano system. There's no more reason to say "1 plus 2 $=$ 3" than there is to say "† plus ‡ $=$ ¿" : both are just ways of saying (in different naming systems) that "0^+ plus $0^{++} = 0^{+++}$" $-$ and in fact, the statement "0^+ plus $0^{++} = 0^{+++}$" has no meaning yet at all because we haven't defined what "plus" means in a Peano system.

We cannot say $2 \cdot 2 = 4$, because (right now) that statement is just a new way of writing $0^{++} \cdot 0^{++} = 0^{++++}$ $-$ which, at the moment, has no meaning at all, because we haven't even defined what it means to "multiply" objects in a

Peano System.

We can, however, use these new names now to record things that we do already know. For example, the axioms for a Peano system now read:

P1) There is one special object named "0" in $\mathcal{P}$
P2) For each object $n \in \mathcal{P}$, there is exactly one object in $\mathcal{P}$ called its successor (and denoted n^+)
P3) 0 is not the successor of any object, that is, $(\forall n \in \mathcal{P})\ n^+ \neq 0$
P4) Different objects have different successors, that is

$$\forall m, \forall n \in \mathcal{P}\ (m \neq n \Rightarrow m^+ \neq n^+)$$

P5) Suppose $A \subseteq \mathcal{P}$. If $0 \in A$ and if

$$(\forall n \in \mathcal{P})\ \ (n \in A \Rightarrow n^+ \in A)$$

then $A = \mathcal{P}$.

Just because of how we named things, statements like these are true:

$0^+ = 1$ (i.e., 1 is the successor of 0) ,
$0^{++} = 1^+ = 2$,
$4^+ = 5$

If we had decided instead to use the naming system in the first row of the table, the following would be true:

œ$^+ =$ †
œ$^{++} =$ †$^+ =$ ‡
ð$^+ =$ €

The theorems we already proved, with the new naming system, can now be written:

Theorem 1 For all $n \in \mathcal{P}$, either $n = 0$ or $n = m^+$ for some $m \in \mathcal{P}$

Corollary 2 If $n \in \mathcal{P}$ and $n \neq 0$, then $n = m^+$ for a unique $m \in \mathcal{P}$.

Theorem 3 For all $n \in \mathcal{P}$, $n \neq n^+$

Theorem 4 *If $n \in \mathcal{P}$, then $n = 0$ or n can be obtained from 0 by applying the successor operation finitely often.*

Corollary 5 *If $m, n \in \mathcal{P}$ and $m \neq n$, then one of m and n can be obtained from the other by applying the successor operation finitely often.*

Defining Arithmetic in a Peano System

<u>**Addition**</u> Let $\mathcal{P}$ be a Peano system (in which we have named the elements $0, 1, 2, ...$).

First, we want to define addition: what does $m + n$ mean? For any given m in $\mathcal{P}$, the definition tells (using P5, the induction axiom) what it means to "add n, on the right, to m."

Definition A Suppose $m \in \mathcal{P}$. Define

i) $m + 0 = m$ and
ii) $\forall n \in \mathcal{P},\ \ (m + n^+) = (m + n)^+$

For any given m, we can use P5) to show that $m + n$ has been defined for every n:

Suppose $m \in \mathcal{P}$. Let $A = \{n \in P : m + n$ is defined$\}$.

By i), $0 \in A$.

If $n \in A$, then $m + n$ is defined. So then $m + n^+$ is also defined because ii) defines $m + n^+$ as the successor of $m + n$ in $\mathcal{P}$. Therefore $n^+ \in A$.

By P5), $A = \mathcal{P}$. •

Example Suppose $m \in \mathcal{P}$. Then

$m + 0 = m$ (by definition Ai)
$m + 1 = (m + 0^+)$ because "1" is the name we assigned to 0^+
$= (m + 0)^+$ by Definition Aii
$= m^+$ by Definition Ai

(*Note: so it turns out, as a result of our definition of addition, that "add 1 to m" is the same thing as "take the successor of m."*)

In particular, if we let $m = 0$, the preceding calculations show that

$$0 + 0 = 0$$
$$0 + 1 = 0^+ = 1$$
$$0 + 2 = (0 + 1)^+ = 1^+ = 2$$

If we let $m = 1$, we see that the preceding calculation shows that $1 + 1 = 1^+$, and the name we assigned to 1^+ is 2 : so $1 + 1 = 2$.

Similarly

$$2 + 1 = 2^+ = 3$$
$$3 + 1 = 3^+ = 4,$$
.
etc.

$($*By convention, let's agree that we may also write* m^{++} *for* $(m^+)^+)$

$$\begin{aligned} m + 2 &= (m + 1^+) && \text{because "2" is the name we assigned to "}1^+\text{"} \\ &= (m + 1)^+ && \text{by Definition Aii} \\ &= (m^+)^+ && \text{by the preceding example} \end{aligned}$$

Letting $m = 1, 2, ...$ gives the specific facts

$$1 + 2 = 1^{++} = 2^+ = 3$$
$$2 + 2 = 2^{++} = 3^+ = 4$$
etc.

Similarly, for any $m, n,$ the recursive definition of addition lets us work backwards "deeper and deeper" until, with a lot of patience $-$ but only finitely many steps $-$ we eventually figure out the sum $m + n$. For example, that $5 + 4 = 9$ $($*give a justification for each step*$)$:

$$5 + 4 = 5 + 3^+ = (5 + 3)^+ = (5 + 2^+)^+ = (5 + 2)^{++}$$
$$= (5 + 1^+)^{++} = (5 + 1)^{+++} = (5 + 0^+)^{+++}$$
$$= (5 + 0)^{++++} = 5^{++++} = 6^{+++} = 7^{++} = 8^+ = 9$$

From the definition of addition (Ai), we know that $m + 0 = m$ for any $m \in \mathcal{P}$. <u>BUT</u> that <u>doesn't</u> mean that we can say $0 + m = m$, because we haven't yet proved that addition in $\mathcal{P}$ is commutative. The next theorem is a first step in that direction.

Theorem 6 $(\forall n \in \mathcal{P})$ $0 + n = n = n + 0$

Proof Let $n \in \mathbb{N}$. We know $n + 0 = n$ by the Definition Ai) $-$ *that equation is included in Theorem 6 as contrast to* $0 + n$. *It is the statement* $(\forall n \in \mathcal{P})$ $0 + n = n$ *that we need to prove.* Let $A = \{n \in \mathcal{P} :\ 0 + n = n\}$.

If $n = 0$, then $0 + 0 = 0 = 0 + 0$, by Definition Ai). So $n = 0 \in A$.

Suppose that $n \in A$. Then

$$\begin{aligned} 0 + n^+ &= (0 + n)^+ && \text{(by Definition Aii, with } m = 0) \\ &= n^+ && \text{(because } n \in A) \end{aligned}$$

Therefore $n^+ \in A$.

By the induction axiom P5), $A = \mathcal{P}$. $\bullet$

To prove that addition is commutative and associative, it's helpful to begin by proving a lemma.

Lemma 7 $(\forall m \in \mathcal{P})(\forall n \in \mathcal{P})\ \ m^+ + n = (m+n)^+ = m + n^+$

Proof We already know that $(m+n)^+ = m + n^+$ by the Definition Aii) $-$ $m + n^+$ *included in Lemma 7 just as contrast to* $m^+ + n$. *It is the statement* $(\forall m \in \mathcal{P})(\forall n \in \mathcal{P})\ \ m^+ + n = (m+n)^+$ *that we need to prove.*

Let $m \in \mathcal{P}$. We need to show that $(\forall n \in \mathcal{P})\ \ m^+ + n = (m+n)^+$

Define $A = \{n \in \mathcal{P} :\ (m+n)^+ = m^+ + n\}$. We want to show that $A = \mathcal{P}$.

$(m+0)^+ = m^+ = m^+ + 0$ (using Definition Ai), so $0 \in A$

Suppose that $n \in A$. We need to show $n^+ \in A -$ that is, we need to show that $(m + n^+)^+ = m^+ + n^+$:

$$
\begin{aligned}
(m^+ + n^+) &= (m^+ + n)^+ && \text{(Definition Aii)}\\
&= (\,(m+n)^+)^+ && \text{(because } n \in A\text{)}\\
&= (\ m + n^+\ )^+ && \text{(Definition Aii)}
\end{aligned}
$$

By P5), $A = \mathcal{P}$. $\bullet$

Theorem 8 a) $(\forall m \in \mathcal{P})(\forall n \in \mathcal{P})(\forall p \in \mathcal{P})\ \ m + (n+p) = (m+n) + p$
(*Addition is* *associative*.)

b) $(\forall m \in \mathcal{P})(\forall n \in \mathcal{P})\ \ m + n = n + m$
(*Addition is* *commutative*.)

Proof a) Suppose m, n be any objects in the Peano system $\mathcal{P}$. We need to show that

$$(\forall p \in \mathcal{P})\ \ m + (n+p) = (m+n) + p$$

Let $A = \{p \in \mathcal{P} :\ m + (n+p) = (m+n) + p\}$. We want to show that $A = \mathcal{P}$.

$$
\begin{aligned}
0 \in A:\ \ m + (n+0) &= m + n && \text{(by Definition Ai)}\\
&= (m+n) + 0 && \text{(by Definition Ai), again)}
\end{aligned}
$$

Suppose, for some p, that $p \in A$. We show that p^+ must be in A.

$$
\begin{aligned}
m+(n+p^{+}) &= m+(n+p)^{+} && \text{(by Definition Aii)}\\
&= (m+(n+p))^{+} && \text{(by Definition Aii, again)}\\
&= ((m+n)+p)^{+} && \text{(because } p\in A)\\
&= (m+n)+p^{+} && \text{(by Definition Aii, again)}
\end{aligned}
$$

Therefore $p^{+} \in A$.

By P5), $A = \mathcal{P}$. •

b) Suppose $m \in \mathcal{P}$. We must show that $(\forall n \in \mathcal{P})\ \ m+n = n+m$.

Let $A = \{n \in \mathcal{P} : \ m+n = n+m\}$.

$m+0=m$, by Definition Ai, and we proved in Theorem 6 that $0+m=m$. Therefore $0 \in A$.

Suppose that $n \in A$. We will show that n^{+} must be in A.

$$
\begin{aligned}
m+n^{+} &= (m+n)^{+} && \text{(by Definition Aii)}\\
&= (n+m)^{+} && \text{(because } n\in A)\\
&= n^{+}+m && \text{(by Lemma 7)}
\end{aligned}
$$

Therefore $n^{+} \in A$.

By P5), $A = \mathcal{P}$. •

Because addition is associative, we often write things like $m+n+p$ *without parentheses, because it doesn't matter whether we interpret this as meaning* $(m+n)+p$ *or* $m+(n+p)$.

Summary: We have defined addition ($+$) in $\mathcal{P}$. We have proved the necessary theorems to compute $m+n$ for any $m,n \in \mathcal{P}$. The addition we created turned out to be commutative and associative, and to have a “neutral” element, $0 :\ m+0=0+m=m$ for all $m \in \mathcal{P}$. In other words (as much as we can see, so far) addition in $\mathcal{P}$ behaves exactly like ordinary addition does in our infromal, intuitive system ω.

In ω, we can also multiply. So now we hope to define a multiplication operation in $\mathcal{P}$ that behaves just like multiplication in ω.

Multiplication

We also want to define multiplication in $\mathcal{P}$. We do that using addition and the successor operation. Then we need to look at some theorems about multiplication behaves in $\mathcal{P}$ and how multiplication is connected to addition.

We could try making a definition like

"$m \cdot n$ means the result of <u>adding</u> m to itself n times."

But this is an inconvenient way to put it because it doesn't give us a precise <u>formula</u> " $m \cdot n =$ ..." to work with: so what do we do?

> *We stop and look for motivation. Think about how multiplication works in the* <u>*informal*</u> *system* ω. *In* ω, $m \cdot 0 = 0$, *and, if you already know how to find* $m \cdot n$, *there is a formula telling you how to find* $m \cdot (n+1)$, *namely*
>
> $$m \cdot (n+1) = m \cdot n + m$$
>
> *We use this fact about the informal system* ω, *to inspire our* <u>*definition*</u> *of multiplication in the formal system* $\mathcal{P}$. *Of course, this makes it likely that multiplication in* $\mathcal{P}$ *will, in fact, act like multiplication in the informal system,* ω. *And that's what we want. We are trying to show how to create, from very simple assumptions, a formal system* $\mathcal{P}$ *that acts like* ω, *so we "build in" what we need to make the finished product be what we want it to be.*

Definition M Suppose $m \in \mathcal{P}$. We define

i) $m \cdot 0 = 0$ and
ii) for any $n \in \mathcal{P}$, $m \cdot n^+ = m \cdot n + m$

(*Sometimes we will just write "mn" for "$m \cdot n$."*)

<u>Exercise</u>: Suppose $m \in \mathcal{P}$. Verify (just as we did for addition) that $m \cdot n$ is defined for all $n \in \mathcal{P}$.

Example For any $m \in \mathcal{P}$,

$$m \cdot 1 = m \cdot 0^+ = m \cdot 0 + m = 0 + m = m$$
$$m \cdot 2 = m \cdot 1^+ = m \cdot 1 + m = m + m$$
$$m \cdot 3 = m \cdot 2^+ = m \cdot 1 + m = (m+m) + m$$

etc.

For example, $3 \cdot 1 = 3$

$3 \cdot 2 = 3 + 3 = 6$ (*using earlier work on addition*)

$$
\begin{aligned}
4 \cdot 3 = 4 \cdot 2^+ &= 4 \cdot 2 + 4 = 4 \cdot 1^+ + 4 \\
&= (4 \cdot 1 + 4) + 4 \\
&= (4 \cdot 0^+ + 4) + 4 = ((4 \cdot 0 + 4) + 4) + 4 \\
&= ((0 + 4) + 4) + 4 = (4 + 4) + 4 \\
&= (\textit{using all the operations for computing sums})... \\
&= \ 8 + 4 = ... = 12
\end{aligned}
$$

The next lemma gives a useful variation on the equations in Definition M. It is an analogue (for multiplication) of Lemma 7 (about addition).

Lemma 9 For all $m, n \in \mathcal{P}$,

a) $0 \cdot m = 0 = m \cdot 0$
b) $m^+ \cdot n = m \cdot n + n$

Proof Suppose $m \in \mathcal{P}$

a) $0 = m \cdot 0$ by Definition Mi). What we need to prove is that $0 \cdot m = 0$

Let $A = \{m \in \mathcal{P} : \ 0 \cdot m = 0\}$

$0 \in A$ because $0 \cdot 0 = 0$ (by Definition Mi)

Suppose $m \in A$. We will show that $m^+ \in A$.

$$
\begin{aligned}
0 \cdot m^+ &= 0 \cdot m + 0 && \text{(by Definition Mii)} \\
&= 0 + 0 && \text{since } m \in A \\
&= 0 && \text{(by Definition Ai)}
\end{aligned}
$$

Therefore $m^+ \in A$.

By P5), $A = \mathcal{P}$. •

b) Let $A = \{n \in \mathcal{P} : \ m^+ \cdot n = m \cdot n + n\}$

$$
\begin{aligned}
0 \in A, \text{ because } m^+ \cdot 0 &= 0 && \text{(by Definition Mi)} \\
&= m \cdot 0 && \text{(by Definition Mi), again)} \\
&= m \cdot 0 + 0 && \text{(by Definition Ai)}
\end{aligned}
$$

Suppose $n \in A$. We show that $n^+ \in A$. To do this, we need to show that

$$m^+ \cdot n^+ = m \cdot n^+ + n^+.$$

$$\begin{aligned}
m^+ \cdot n^+ &= m^+ \cdot n + m^+ && \text{(by Definition Mii)} \\
&= (m \cdot n + n) + m^+ && \text{(because } n \in A\text{)} \\
&= m \cdot n + (n + m^+) && \text{(by Theorem 8: addition is associative)} \\
&= m \cdot n + (n + m)^+ && \text{(by Definition Aii)} \\
&= m \cdot n + (m + n)^+ && \text{(by Theorem 8; addition is commutative)} \\
&= m \cdot n + (m + n^+) && \text{(by Definition Aii)} \\
&= (m \cdot n + m) + n^+ && \text{(by Theorem 8: addition is associative)} \\
&= m \cdot n^+ + n^+ && \text{(by Definition Mii)}
\end{aligned}$$

Therefore $n^+ \in A$.

By P5, $A = \mathcal{P}$. •

We can now prove a connection between addition and multiplication (the distributive rule) and see that multiplication is associative and commutative. For convenience, we agree to write mn for $m \cdot n$.

Theorem 10 $(\forall m \in \mathcal{P})(\forall n \in \mathcal{P})(\forall p \in \mathcal{P})$

a) $m(n + p) = mn + mp$ ($\cdot$ *and* $+$ *are connected by the distributive rule*)
b) $m(np) = (mn)p$ (*Multiplication is associative.*)
c) $mn = nm$ (*Multiplication is commutative.*)

Proof a) The proof of a) is an assigned problem in the homework. We assume a) in the arguments below.

b) Suppose $m, n \in \mathcal{P}$. We need to show that $(\forall p \in \mathcal{P})\ \ m(np) = (mn)p$

Let $A = \{p \in \mathcal{P} : m(np) = (mn)p\}$. We want to show $A = \mathcal{P}$.

$0 \in A$ since
$$\begin{aligned}
m(n \cdot 0) &= m \cdot 0 && \text{(by Definition Mi)} \\
&= 0 && \text{(by Definition Mi, again)} \\
&= (mn) \cdot 0, && \text{(by Definition Mi, again)}
\end{aligned}$$

Suppose, for some p, that $p \in A$. Then

$$\begin{aligned}
m(np^+) &= m(np + n) && \text{(by Definition Mii)} \\
&= m(np) + mn && \text{(by part a) of this theorem: the}
\end{aligned}$$

distributive rule)

$= (mn)p + mn$ (because $p \in A$)

$= (mn)p^+$ (by Definition Mii)

Therefore $p^+ \in A$.

By P5, $A = \mathcal{P}$. •

c) Suppose $m \in \mathcal{P}$. We need to show that $(\forall n \in \mathcal{P})\ mn = nm$

Let $A = \{n \in \mathcal{P} : mn = nm\}$. We want to show $A = \mathcal{P}$.

$0 \in A$ because $m \cdot 0 = 0 = 0 \cdot m$ (by Lemma 9)

Suppose, for some n, that $n \in A$. Then

$mn^+ = mn + m$ (by Definition Mii)

$= nm + m$ (because $n \in A$)

$= n^+m$ (by Lemma 9)

Therefore $n^+ \in A$.

By P5, $A = \mathcal{P}$ •

Because multiplication is associative, we often write things like mnp without parentheses, because it doesn't matter whether we intended $(mn)p$ or $m(np)$.

Example An earllier example (with $m = 3$) showed that $m \cdot 3 = (m + m) + m$. We could also get this fact from <u>addition and the distributive law</u>:

$$(m + m) + m = (m \cdot 1 + m \cdot 1) + m \cdot 1$$

$$= m \cdot (1 + 1) + m \cdot 1 = (m \cdot 2) + m \cdot 1 = m(2 + 1) = m \cdot 3.$$

By the <u>commutative</u> law for multiplication, we can now say also that

$$m \cdot 3 = (m + m) + m = 3 \cdot m.$$

In the proofs that follow, we will now use the definitions of $+$ and $\cdot$ more freely (without always citing an explicit justification for each and every step). We will also freely use that multiplication are associative and commutative, and that the distributive law is true in $\mathcal{P}$. In some arguments, such as the proof of part c) of the following theorem, we use of results previously proven and don't need to use an induction in the argument.

Theorem 11 Suppose $m, n, c \in \mathcal{P}$.

a) If $m \neq 0$, then $m + n \neq 0$.

b) (Cancellation for $+$) If $m + c = n + c$, then $m = n$.
(If $c + m = c + n$, then $m + c = n + c$, so $m = n$. The theorem tells us that we can "cancel c on the left", too.)

c) If $m \neq 0$ and $n \neq 0$, then $mn \neq 0$.

Note: We already proved in Theorem 8b) that addition is commutative. Therefore it doesn't matter in part a) whether the nonzero term, m, is on the left or the right: $m + n = n + m$: in words, 11a) merely says that the sum of two obejcts from $\mathcal{P}$ is not 0 if one of the objects is not 0.

Suppose $m + n = 0$. What can we conclude in $\mathcal{P}$?

Proof a) Suppose $m \neq 0$. Let $A = \{n \in \mathcal{P} : m + n \neq 0\}$.

$0 \in A$ because $m + 0 = m \neq 0$.

Suppose that $n \in A$. Then $m + n^+ = (m + n)^+ \neq 0$ (using P3).
Therefore $n^+ \in A$. By P5, $A = \mathcal{P}$. $\bullet$

b) This proof is an assigned exercise in the homework.

c) Suppose $m \neq 0$ and $n \neq 0$. We know that $n = k^+$ for some k (by Theorem 1), so $mn = mk^+ = mk + m$. Since $m \neq 0$, we conclude that $mn \neq 0$ (using part a) of this theorem). $\bullet$

(*Note: Part c) is done without using induction (P5). <u>However</u>, the proof uses other results (such as Theorem 1) that <u>were</u> proved using the induction axiom P5.*

Example For short, we can agree to write "n^2" for "$n \cdot n$", n^3 for "$(n \cdot n) \cdot n$", etc. Show that $(n+1)(n+2) = n^2 + 3n + 2$. (*Justify each step! Be sure that each "arithmetic calculation" is one that we justified.*)

$$\begin{aligned}(n+1)(n+2) &= ((n+1) \cdot n) + (n+1) \cdot 2 = (n \cdot (n+1)) + 2 \cdot (n+1) \\ &= (n^2 + n \cdot 1) + (2 \cdot n + 2 \cdot 1) = (n^2 + n) + (2n + 2) \\ &= (n^2 + (n + 2n)) + 2 \\ &= (n^2 + n \cdot (1+2)) + 2 \\ &= (n^2 + n \cdot 3) + 2 \\ &= (n^2 + 3 \cdot n) + 2 \\ &= n^2 + 3 \cdot n + 2\end{aligned}$$

(*Be sure you can justify each step*)

On the surface, it looks like we have shown, without induction, that

$$(\forall n \in \mathcal{P}) \ (n+1)(n+2) = n^2 + 3n + 2$$

In fact, nearly every step in the calculations is justified by a theorem whose proof <u>did</u> use induction.

The truth is that a proof for <u>any</u> statement of the form

$$(\forall n \in \mathcal{P}) \ P(n)$$

<u>must</u> depend on the induction axiom P5 (either in the proof itself, or in the proofs of earlier theorems that are used in the proof).

Defining an Ordering Relation in a Peano System

Finally, we can introduce an "ordering " (denoted by $\leq$) in $\mathcal{P}$ with another definition.

Definition O Suppose $m, n \in \mathcal{P}$. We say $m \leq n$ iff $(\exists c \in \mathcal{P})\,(m + c = n)$.
We write $m < n$ iff $m \leq n$ and $m \neq n$.

Note: We also write $m \leq n$ as $n \geq m$: the two relations are understood to mean the same thing.
Similarly, the relations $m < n$ and $n > m$ are understood to mean the same thing.

Example For each $m \in \mathcal{P}$:

$0 + m = m$ so, by Definition O, $0 \leq m$

$m + 0 = m$ so, by Definition O, $m \leq m$.

Theorem 12 For all $m, n, p \in \mathcal{P}$:

a) $m \leq m$
b) if $m \leq n$ and $n \leq p$, then $m \leq p$.
c) if $m \leq n$ and $n \leq m$, then $m = n$.

Proof a) See the example, above.

b) If $m \leq n$, there is a c such that $m + c = n$, and
if $n \leq p$, there is a d such that $n + d = p$.
Therefore $m + (c + d) = (m + c) + d = n + d = p$,
so $m \leq p$.

c) If $m \leq n$, there is a c such that $m + c = n$, and
if $n \leq m$, there is a d such that $n + d = m$.

Since $m + c = n$,
$(m + c) + d = n + d = m$, so
$m + (c + d) = m = m + 0$.
Theorem 11b) lets us cancel the m and get $c + d = 0$.
But then, by Theorem 11a), $c = d = 0$.

Therefore $n = m + c = m + 0 = m$. •

Theorem 13 $(\forall m \in \mathcal{P})\,(\forall n \in \mathcal{P})\;(\,m \leq n$ or $n \leq m)$

Proof Let $m \in \mathcal{P}$. For this m, we need to show that $(\forall n \in \mathcal{P})\;(\,m \leq n$ or $n \leq m)$. Let $A = \{n \in \mathcal{P} :\; m \leq n$ or $n \leq m$ is true$\}$. We will show that $A = \mathcal{P}$.

The example above shows that $0 \leq m$, so $0 \in A$.

Suppose that $n \in A$. (*Since A is defined by a statement using "or", there are two cases to consider.*)

i) If $m \leq n$, then $\exists c \in \mathcal{P}$ such that $m + c = n$. In that case, $m + c^+ = (m+c)^+ = n^+$ so $m \leq n^+$ and therefore $n^+ \in A$.

ii) If $n \leq m$, then $\exists c \in \mathcal{P}$ such that $n + c = m$.

If $c = 0$, then $n = m$, so $m + 1 = m^+ = n^+$, which means that $m \leq n^+$, so $n^+ \in A$.

If $c \neq 0$, then c has a predecessor d in $\mathcal{P}$: $c = d^+$.
Then $n^+ + d = n + d^+$ (by Lemma 7)
$= n + c = m$, so $n^+ \leq m$
and therefore $n^+ \in A$.

In both cases, $n^+ \in A$.

Therefore, by P5, $A = \mathcal{P}$. $\bullet$

Note: Corollary 5 in the "Peano Systems" notes was printed in italics because it seemed to involve some questionable reasoning. Theorem 13 is a correct, rigorous version of what Corollary 5 was trying to say: if $m \neq n$, then "one of these two objects in $\mathcal{P}$" = "the other object + c" and, as we have seen in several examples, "adding c" turns out to be the same as repeated applications of the successor operation.

Corollary 14 $(\forall m \in \mathcal{P})(\forall n \in \mathcal{P})\;(\,m < n$ or $m = n$ or $m > n)$

Proof By Theorem 13**,** we know $m \leq n$ or $m \geq n$. If $m \neq n$, then (by definition of $<$), we know that $m < n$ or $m > n$. $\bullet$

Theorem 15 (Cancellation for multiplication) If $m, n, p \in \mathcal{P}$ and $p \neq 0$ and $mp = np$, then $m = n$.

(*Since multiplication is commutative, if $pm = pn$, then $mp = np$ so $m = n$. Therefore theorem tells us that we can also cancel a nonzero factor of p on the left.*)

Proof Homework exercise.

Finally, we want to check that the "order relation" $\leq$ interacts nicely with addition and multiplication (*just as* $\leq$, +, *and* $\cdot$ *interact nicely in the informal system* ω.) For example,

Theorem 16 Suppose $m, n, p \in \mathcal{P}$ and that $m \leq n$. Then

a) $m + p \leq n + p$

b) $mp \leq np$.

Proof a) Since $m \leq n$, there is a $c \in \mathcal{P}$ such that $m + c = n$. Then

$(m + c) + p = n + p$, so
$(m + p) + c = n + p$, so
$m + p \leq n + p$. •

b) This is an assigned problem in the homework.