Three Important Facts About $\mathbb{N}$

There are three important statements about the set of natural numbers, $\mathbb{N}$, that turn out to be equivalent to each other. As mentioned in class, minor variations of each one are also possible.

I. PMI (Principle of Mathematical Induction) Suppose $S \subseteq \mathbb{N}$.

If a) $1 \in S$, and
b) $\forall n\,(n \in S \Rightarrow n+1 \in S)$
then $S = \mathbb{N}$

II. PCI (Principle of Complete Induction) Suppose $S \subseteq \mathbb{N}$.

If $\forall n\,(\,\{k : k < n\} \subseteq S \;\Rightarrow n \in S\}\,)$ $(*)$
then $S = \mathbb{N}$

(Note: In class, I stated PCI as follows:

If i) $1 \in S$, *and*
ii) $\forall n > 1 \;\; (\{k : k < n\} \subseteq S \;\Rightarrow n \in S\,)$
then $S = \mathbb{N}$

When $n = 1$, $\{k : k < 1\} = \emptyset$ *and, of course,* $\emptyset \subseteq S$ *is true. So, if the conditional statement* $(*)$ *is true, it follows that* $1 \in S$. *There's no logical requirement to write separately (as I did in class) that you need to check* $1 \in S$ *when using CPI; writing that is redundant.*

But when you use CPI to prove a specific theorem $(\forall n)P(n)$, *you do need to verify that* $(*)$ *is true for all* n *(including* $n = 1$*). And to do that, the case* $n = 1$ *is often special: the reason is that since* $\{k : k < 1\} = \emptyset$*, you have no "previously proved propositions* $P(k)$ *for* $k < 1$*" to use in proving* $P(1)$*, so* $P(1)$ *must be checked "from scratch."). In practice, you often need to start out by checking the* $n = 1$ *case separately anyway.*

To summarize: Listing "$1 \in S$*" (as in class) as part of the statement of PCI is technically redundant, but the redundancy is harmless and is often a useful reminder not to forget about the case* $n = 1$*.)*

III. WOP (Well-Ordering Principle) Suppose $A \subseteq \mathbb{N}$

If $A \neq \emptyset$
then A contains a smallest element.

For now, these three statements are treated as intuitively clear statements about $\mathbb{N}$ which can used to prove more complicated facts. Later, we will see later that, in fact, that they are true about $\mathbb{N}$ because they are "built into $\mathbb{N}$" when $\mathbb{N}$ is constructed from set theory.

Even though PMI, PCI, and WOP are equivalent, sometimes one of them is more convenient to use than the others. We have seen examples using each of them.

Perhaps you feel that WOP is easier to believe than the others. But (to repeat), PMI, PCI and WOP are equivalent statements about $\mathbb{N}$: you must believe all three of the statements, or none of them.

To prove that PMI, PCI, and WOP are equivalent, we will give three arguments: :

PMI $\Rightarrow$ PCI   PCI $\Rightarrow$ WOP   WOP $\Rightarrow$ PMI.

Each proof follows on a separate page.

Prove PMI $\Rightarrow$ PCI

Suppose $S \subseteq \mathbb{N}$ and assume that PMI is true. We want to prove PCI:

$$\begin{array}{ll} \text{If} & \forall n\,(\,\{k : k < n\} \subseteq S \;\Rightarrow n \in S\}\,) \qquad (*) \\ \text{then} & S = \mathbb{N} \end{array}$$

To do this, assume $(*)$: $\forall n\,(\,\{k : k < n\} \subseteq S \;\Rightarrow n \in S\})$.
We need to show that $S = \mathbb{N}$.

> (*Strategy: We will show that for every n, $\{1, 2, ..., n\} \subseteq S$. If that is true, then $n \in S$ for every n, which tells us that $\mathbb{N} \subseteq S$. Since we already know $S \subseteq \mathbb{N}$, we will conclude that $S = \mathbb{N}$.*
>
> *Let's carry out the strategy.*)

a) Since $\{k : k < 1\} = \emptyset \subseteq S$, (*) gives that $1 \in S$, so $\{1\} \subseteq S$.

b) Assume that for some n, we have $\{1, 2, ..., n\} \subseteq S$ – that is, assume $\{k : k < n+1\} \subseteq S$. By (*), we conclude that $n+1 \in S$. Therefore $\{1, 2, ..., n, n+1\} \subseteq S$.

> *Summarizing so far:* $1 \in S$ *and* $($ *if* $\{1, 2, ..., n\} \subseteq S$, *then* $\{1, 2, ..., n+1\} \subseteq S$ $)$.

Using our assumption that PMI is true, we conclude that for $\forall n$ $\{1, 2, ..., n\} \subseteq S$. Therefore $\mathbb{N} \subseteq S$, so (*as outlined in the strategy*), $S = \mathbb{N}$. •

Prove PCI $\Rightarrow$ WOP

Assume PCI is true. We want to prove WOP:

Suppose $A \subseteq \mathbb{N}$.

If $A \neq \emptyset$,
then A contains a smallest element.

(*Strategy: We will the contrapositive of WOP instead. We assume that A contains no smallest element, and prove that $A = \emptyset$.*

Let's carry out the strategy.)

Assume A contains no smallest element, and let $S = \mathbb{N} - A$.

$(\{k : k < 1\}) \Rightarrow 1 \in S$. (Since A has no smallest element, $1 \notin A$ and therefore $1 \in S$. Since $1 \in S$ is true, the conditional statement is true.)

Suppose, for some $n > 1$, $\{k : k < n\} = \{1, 2, ..., n-1\} \subseteq S$. Then $1, 2, ..., n-1$ are not in A. Therefore $n \notin A$ (because n would be the smallest element in A). Therefore $n \in S$.

Summarizing so far: If $\{k : k < n\} \subseteq S$, then $n \in S$.

Using PCI, we conclude $S = \mathbb{N}$. But $S = \mathbb{N} - A$, so therefore $A = \emptyset$. (*See the strategy*) •

Prove WOP $\Rightarrow$ PMI

Assume that WOP is true. We want to prove PMI:

Suppose $S \subseteq \mathbb{N}$.

If $1 \in S$ and $\forall n (n \in S \Rightarrow n+1 \in S)$
then $S = \mathbb{N}$.

(Strategy: We will prove an equivalent statement: PMI is equivalent to

If $1 \in S$ *and* $S \neq \mathbb{N}$, *then* $\sim (\forall n)(n \in S \Rightarrow n+1 \in S)$

which, in turn, is equivalent to

If $1 \in S$ *and* $S \neq \mathbb{N}$, *then* $(\exists n)\,(n \in S \;\wedge\; n+1 \notin S)$

We are using, from logic, that

$P \wedge Q \Rightarrow R$ is equivalent to $P \wedge \sim R \Rightarrow \sim Q$.

Let's carry out the strategy.)

Suppose $1 \in S$ and $S \neq \mathbb{N}$. Then $\mathbb{N} - S \neq \emptyset$, so by WOP there is a smallest natural number, call it k, in $\mathbb{N} - S$.

We know $k \neq 1$ (since $1 \in S$) so $n = k - 1 \in \mathbb{N}$. Then $n \in S$ (since k was the smallest natural number not in S). Therefore $n \in S$ but $n + 1 = k \notin S$. (*See the strategy*). •