The Basics of Set Theory

Introduction

Every math major should have a basic knowledge of set theory. The purpose of this chapter is to provide some of that basic information.

Sets provide a useful vocabulary in many situations. They are a handy language for stating interesting results in all areas of mathematics – for example,

"A group is a set such that..." or
"A basis for the vector space V is a set $\mathcal{B}$ of vectors such that ...".

Set theory had its origins in work done by Georg Cantor (during the late 19th century) on certain kinds of infinite series called Fourier series. However sets are not just a tool: like many other mathematical ideas, "set theory" has grown into a fruitful research area of its own.

Moreover, on the philosophical side, most mathematicians accept set theory as a foundation for mathematics – that is, the notions of "set" and "membership in a set" can be used as the most primitive ideas in terms of which all mathematical objects and ideas can be defined. From this point of view, everything in mathematics (numbers, relations, functions, ...) is a set. To put it extreme way, most mathematicians believe (when pressed to the bottom line) that "mathematics can be thought of as just a part of set theory." As this course goes on, we'll get some idea of why this point of view is reasonable.

So, you ask, what is a set? There are several different ways to try to answer. Intuitively – and this is good enough for most of our purposes – a set is a collection of objects, called its *elements* or *members*. For example, we can talk about "the set of United States citizens" or "the set of all real numbers." The idea seems clear enough. However, we have not really given a satisfactory definition of a set – it seems circular (after all, what is a "collection" if not just another way of saying "set"?).

In the beginning, writers tried to give sharp definitions for "set," just as Euclid tried to give definitions for such things as "straight line" (= "a line which lies evenly with the points on itself"). Of course Euclid's definitions really wouldn't clarify much to somebody who didn't already have ideas about straight lines. Similarly, the old attempts to "define" a "set" were really not very satisfying. For example, according to Cantor,

> *Unter einer Menge verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten in unserer Anschauung oder unseres Denkens (welche die Elemente von M genannt werden) zu einem Ganzen* [By a set we are to understand any collection into a whole M of definite and separate objects (called the elements of M) of our perception or thought.] (*German seems to be a good language for this kind of talk.*)
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> More compactly, Felix Hausdorff, around 1914, stated that a set is "a plurality thought of as a unit."

At this stage, we have several options.

i) We can use our intuitive, informal notion of a set and go on from there, ignoring any more subtle issues – just as we might not worry about a definition for "point" and "line" in beginning to study geometry.

ii) We can try to give a formal definition of "set" in terms of some other mathematical objects. We would be assuming, implicitly, that these other objects are even "more fundamental" or "clearer" for our use as the foundational objects.

iii) We can take the notions of "set" and "set membership" as "ground zero" – that is, as primitive undefined terms. We won't even ask what sets "really are." We just write down some rules (axioms) about how these things we call "sets" behave and proceed from there, in accordance with these rules, to prove new results and define new objects – eventually building up more and more of mathematics.

The first approach is sometimes called naive (or "informal") set theory. Here, the word "naive" merely refers to the starting point; it does not mean "simplistic" – naive set theory actually can get very complicated. Historically, set theory began along these lines.

The second option certainly is a logical possibility but it seems to be one that few if any mathematicians follow. In the work *Principia Mathematica* (mentioned in class), Russell and Whitehead tried to use what we'd call "symbolic logic" as a foundation even more basic than set theory.

The third option would take us into the subject called "axiomatic set theory." Although an enormous amount of interesting and useful naive set theory exists, almost all research work in set theory nowadays requires using this axiomatic approach (as well as a healthy does of mathematical logic).

As a practical matter, we are going to take the naive approach. For one thing, the axiomatic approach is not worth doing if it isn't done carefully, and that is a whole course in itself. Moreover, axiomatic set theory isn't much fun unless you have learned enough naive set theory to appreciate why an axiomatic approach would be important. It's more interesting to try to make things absolutely precise after you have a good overview. (People were aware of lots a things about geometry before Euclid did his axiomatization.)

As we go along, however, we will also make some side comments in the lectures and notes about the axiomatic approach just to provide some perspective. It is the axiomatic approach, when very carefully worked out, that actually provides a foundation for mathematic in set theory. In this course, we at least want some glimpses of how the foundation is laid.

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Preliminaries and Notation

Informal Definition 2.1 A set is a collection of things called its elements (or members). If A is a set and x is an element of A, we write $x \in A$. If x is not a member of A, we write $x \notin A$.

One way to write a small set is to list its members inside curly braces: $A = \{1, 2, 3\}$ is the set having the numbers $1, 2, 3$ as its members.

As the informal definition implies, we may also use the word "collection" (or other similar words such as "family") in place of "set." Sometimes this is just for variety; sometimes it serves informally to emphasize some point – for example, we might refer to a set whose elements are other sets as a "collection of sets" or a "family of sets," rather than a "set of sets."

> In the same vein, using a capital "A" for a set but a lower case letter like "x" for a member of A is just a notational device to help us (psychologically) keep track of things. We might also use other letter styles to help. For example if A, B, and C are sets, we might use a script letter like $\mathcal{B}$ to denote a family (set) of sets : $\mathcal{B} = \{A, B, C\}$ and lower case letters like x, y, z for the members of the set A.
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> However, there's no logical necessity controlling the notation. If we want to, we can use (say) only lower case letters for everything. We could, for example, have sets x, y, z. It might that w is an element of x, that is, $w \in x$. We might then form a new set $v = \{\{x, y\}, \{x, z\}, \{y, z\}\}$ – so that v is a set of sets of sets. It's important to be able to think at this level of abstraction sometimes, but you can see how the use of different cases and fonts can be a useful device. You probably also agree that referring to v as a "family of collections of sets" rather than a "set of sets of sets" helps keep things straight – even though the phrases have identical meanings. .

We can describe sets in a couple of different ways:

By listing the elements – most useful when the set is a small finite set or an infinite set whose elements can be referred to using "..."

For example,

$A = \{1, 2\}$

$\mathbb{N} = \{1, 2, 3, ...\}$ the set of natural numbers

> Some people include "0" in what they call the set of natural numbers. Whether you do or don't is really just a convention about how you name things. When you are reading any particular math book, you always have to be sure how the author is using certain symbols because there are small variations like this.

$\omega = \{0, 1, 2, ...\}$ the set of whole numbers

$\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ the set of integers

By abstraction, that is, by suing some property to describe exactly what elements are in the set. We do this by writing something like $\{x : x \text{ has a certain property}\}$.

For example,

$\mathbb{R} = \{x\!: x \text{ is a real number}\}$	the set of all real numbers
$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{Z} \text{ and } q\ \neq 0\}$	the set of rational numbers
$\mathbb{P} = \{x : x \in \mathbb{R} \text{ and } x \notin \mathbb{Q}\}$	the set of irrational numbers

Following this procedure, we might write down things like

$$\{x : x \in \mathbb{R} \text{ and } x^2 = -1\} \text{ and } \{x :\ x \in \mathbb{R} \text{ and } x \neq x\}.$$

Of course, no real number is actually a member of either set – both sets are empty. The empty set is usually denoted by the symbol $\emptyset$ (which, by the way, is a Danish letter, not a Greek phi (Φ or ϕ)). It is occasionally also denoted by { } . The "empty set" is also known by the more British name "null set."

It might seem odd to allow an empty set and even give it with a special symbol, but the alternative would be to say that some expressions like $\{x :\ x \in \mathbb{R} \text{ and } x^2 = -1\}$, which look perfectly reasonable are, in fact, not sets at all. Even worse, if we did not allow the possibility of an empty set, then we might be uncertain whether some things are sets because were uncertain whether there are any objects in the collection. For example, do you know whether

$$\{x : x \in \mathbb{Q} \text{ and } x = \alpha^\beta, \text{ where } \alpha \text{ and } \beta \text{ are irrational}\}$$

actually contains any members ?

Of course, our informal sets may contain any objects as elements. But in mathematics we are not likely to be interested in sets of aardvarks. We will only use sets that contain various mathematical objects. For example, a set of functions

$$\{\, f :\ \ f \text{ is a continuous real-valued function defined on the closed interval } [a, b]\,\}$$

or a set of sets such as

$$\{\,\{1\}, \{1{,}2\}\,\} \qquad \text{or } \{\emptyset\}, \qquad \text{or } \{\emptyset, \{\emptyset\}\}.$$

Of course, if "everything in mathematics is a set," then (at the "bottom line") all sets in mathematics are sets whose members are other sets (because what else is there?).

We say that A is a subset of B , written $A \subseteq B$, provided each element of A is also a member of B. The more formal definition is:

Definition $A \subseteq B$ if $(\forall x)\ (x \in A \Rightarrow x \in B)$
(*Remember: it's customary of write "if" in a definition, but in as statement which is an "announced" to be a definition, the "if " really means "iff."*)

We say that two sets are equal, $A = B$ when A and B have precisely the same elements. The more formal definition is:

Definition $A = B$ if $(A \subseteq B\ \wedge\ B \subseteq A)$

Clearly, this is equivalent to saying: $(\forall x)\ (x \in A \Leftrightarrow x \in B)$

If $A \subseteq B$ but $A \neq B$ we say A is a proper subset of B.

For example, $\{1, 2\} = \{2, 1\}$ (order doesn't matter in writing down the elements in a set)
$\{x, y\} = \{y, x\}$
$\{x, x\} = \{x\}$

Two sets whose descriptions appear quite different may turn to be equal when you look more carefully For example, you can easily check that

$$\{x : x \in \mathbb{R} \text{ and } x^5 + 5x^4 - 29x^3 - 109x^2 - 8x + 140 = 0\} \ = \ \{-7, -2, 1, 5\}$$

Take a look at each of the following true statements to be sure the notation is clear:

$$\mathbb{N} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

$$x \in A \text{ iff } \{x\} \subseteq A$$

$$\emptyset \neq \{\emptyset\} \qquad \emptyset \subseteq \{\emptyset\} \qquad \emptyset \in \{\emptyset\}$$

Notice that $\emptyset \neq \{\emptyset\}$. The set on the left is empty, while the set on the right has one element, namely the set $\emptyset$. This might be clearer with the alternate notation: $\{\} \neq \{\,\{\}\,\}$. The set on the left is like empty paper bag, but the set on the right is like a bag with an empty bag inside.

Examples

$$\emptyset \subseteq \emptyset \qquad \emptyset \notin \emptyset \qquad \emptyset \subseteq A \text{ for any set } A$$

$\emptyset \in \{\emptyset\} \in \{\{\emptyset\}\}$, but $\emptyset \notin \{\{\emptyset\}\}$
(so $A \in B \in C$ doesn't imply $A \in C$)

If $A \subseteq B \subseteq C$, then $A \subseteq C$

We define the power set of a set A, denoted $\mathcal{P}(A)$, to be the set of all subsets of A.
In symbols, $\mathcal{P}(A) = \{B : B \subseteq A\}$.

Since $\emptyset$ is a subset of every set, we have $\emptyset \in \mathcal{P}(A)$ for every set A. And, since $A \subseteq A$ for any set A, we also have $A \in \mathcal{P}(A)$ for every set A.

So, for example,

$$\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

$$\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

$$\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$$

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

These examples suggest if A has n elements, then $P(A)$ has 2^n elements (that is, A has 2^n subsets). We can prove more carefully this later when we talk about "proofs by induction." But you should be able now to convince yourself, intuitively, that it's true.

(*If I flip a penny 2 times, how many outcomes are possible:* (H,H), (H,T), (T,H), (T,T). *What if I flip the penny* n *times? Why is this "the same" as asking "how many subsets does a set with* n *elements have?"*)

Paradoxes

The naive approach to sets seems to work fine until someone really starts trying to cause trouble. The first person to do this was Bertrand Russell who, around 1902, created Russell's Paradox:

It makes sense to ask whether a set might be one of its own members, that is, for a given set A, to ask whether $A \in A$ is true or false. For the simple sets that you first think about, this statement is clearly false. For example, $\{1,2\} \notin \{1,2\}$. But you hesitate for a moment if

$$A = \{\, \emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}, \{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \{\{\emptyset\}\}\}\}\,\}$$

Might A be a member of itself. (*A little thought about counting { 's and } 's shows this couldn't be true*.) Now suppose we had a similar looking infinite set of sets of sets of sets of...? Could it happen that $A \in A$? Whatever the answer, it makes sense to ask the question.

According to the naive approach to sets we've been using, we can create down a new set $\mathfrak{A}$ (of sets) by writing $\mathfrak{A} = \{A : A \notin A\}$, so that $\mathfrak{A}$ is the "set of all sets which are not members of themselves."

We can then ask, for this new set $\mathfrak{A}$, whether $\mathfrak{A} \in \mathfrak{A}$ is true or false.

If $\mathfrak{A} \in \mathfrak{A}$, then it must be that $\mathfrak{A}$ satisfies the membership requirement for being in $\mathfrak{A}$, which is that $\mathfrak{A} \notin \mathfrak{A}$. So if $\mathfrak{A} \in \mathfrak{A}$, then $\mathfrak{A} \notin \mathfrak{A}$. That's not possible.

On the other hand, if $\mathfrak{A} \notin \mathfrak{A}$, then $\mathfrak{A}$ meets the membership requirement getting into $\mathfrak{A}$, so $\mathfrak{A} \in \mathfrak{A}$. That's not possible either.

Thus, each of the only two possible assumptions about the set $\mathfrak{A}$ (that $\mathfrak{A} \in \mathfrak{A}$ or $\mathfrak{A} \notin \mathfrak{A}$) leads to a contradiction! It seems like there's a contradiction built right into our set theory.

Russell's Paradox illustrates why we have to be just a little more careful: by using the method of abstraction to write down sets too casually, we can dig ourselves into a hole To avoid a built-in contradiction, we somehow don't want to be allowed to call $\mathfrak{A}$ a set. One way to accomplish that, in practice, is to insist that whenever we define a set by abstraction, we only form subsets of already existing sets. That is, in defining a set by abstraction, we always write $\{x$: $x \in U$ and ... $\}$, a statement which we might also write as $\{x \in U :$ ... $\}$.)

The result is the we are defining a subset of some set U that we already have at hand. Since the preceding definition of $\mathfrak{A}$ doesn't follow this form, we will no longer be forced to think that $\mathfrak{A}$ is a set. This lets us avoid Russell's paradox. Watch what happens if we try to recreate Russell's paradox now:

Suppose U is a some set and (according to our new rule about defining sets) we define

$$\mathfrak{A} = \{A \in U\colon A \notin A\}.$$

The dilemma has vanished:

If $\mathfrak{A} \in \mathfrak{A}$, then $\mathfrak{A} \in U$ and $\mathfrak{A} \notin \mathfrak{A}$ which is impossible.

If $\mathfrak{A} \notin \mathfrak{A}$, then $\mathfrak{A}$ does not meet the membership requirements for getting into $\mathfrak{A}$ $-$ which *now* means that either $\mathfrak{A} \notin U$ or $\mathfrak{A} \in \mathfrak{A}$. Since $\mathfrak{A} \in \mathfrak{A}$ is not possible, we merely conclude that $\mathfrak{A} \notin U$, and we can live with that : it's not a contradiction.

Russell's Paradox has the same "flavor" as lots of "self-referential" paradoxes in logic. For example, some books in the library mention themselves – in the preface of a book, for example, the author might say, "In this book, I shall ..." Other books make no mention of themselves. Suppose Olin Library wants to make a book listing all books that do not mention themselves. Should this new book list itself? That is Russell's Paradox. The common-sense resolution of the paradox is to reply something like "Look, what the library really meant was that they want to make a list of all books already in their collection and which do not mention themselves— that is, in forming the new book, one is restricted to considering examining only those books in some preexisting collection U. With this additional qualification, the paradox doesn't come up.

In doing everyday mathematics, we usually don't have to worry about the issue of paradoxes. Almost always, when we form a new set, we have (at least implicitly, in the back of our minds) some larger set U (a "universe") and we are defining some subset of that universe. Therefore, indulging in a bit of sloppiness, we may sometimes write such things as $\{x : \ ... \}$ rather than the more correct
$\{x \in U : \ ... \}$ simply because the set U could be supplied on demand, and the notation is simpler.

There are also other kinds of paradox that can arise from defining sets too casually, but a math major (or even a research mathematician, in day-to-day work)isn't like to bump into them. However, one of the reasons to develop axiomatic set theory carefully is to avoid paradoxes.

Operations on Sets

We want to be able to form new sets from old ones. The simplest operations to do this are union and intersection. The union of two sets A, B is the set $A \cup B$ consisting of all elements in one or the other. The intersection $A \cap B$ of the two sets is the set of all elements belonging to both. More formally

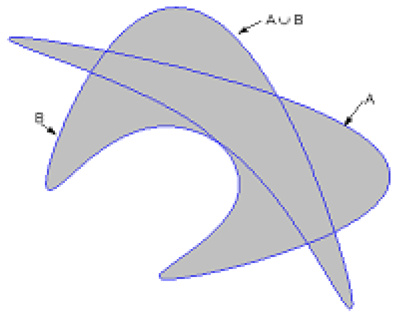
$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x : x \in A \text{ and } a \in B\}.$$

Note: When we discussed the logical meaning of "or", we said that mathematicians use "or" in an inclusive sense. Thus, "$x \in A$ or $x \in B$" means "$x \in A$ or $x \in B$ or both."

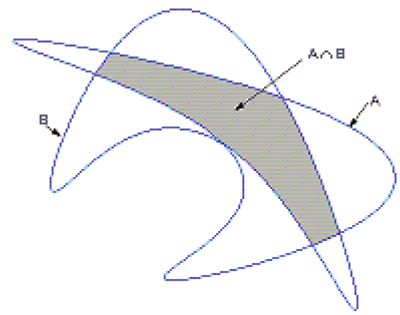
Examples 4.1 $\{1,2\} \cup \{2,3\} = \{1,2,3\}$ $\{1,2\} \cap \{2,3\} = \{2\}$

$\mathbb{P} \cup \mathbb{Q} = \mathbb{R}$ $\mathbb{P} \cap \mathbb{Q} = \emptyset$

The idea of the union and intersection of two sets can be illustrated schematically with "Venn diagrams" :



$A \cup B$



$A \cap B$

Note: For those who are being really careful, our discussion of paradoxes gives you a right to say that the definition of union of A and B should read

$$\{\underline{x \in X} : \ x \in A \text{ or } x \in B\}$$

and then ask "What is X?"

In practice, the sets we unite are always subsets of some larger set and we do not have to worry. (When we write $\mathbb{P} \cup \mathbb{Q} = \{x : x \in \mathbb{P} \text{ or } x \in \mathbb{Q}\}$, we know in our minds that $\mathbb{P} \subseteq \mathbb{R}$ and $\mathbb{Q} \subseteq \mathbb{R}$ so that could write more precisely that $\mathbb{P} \cup \mathbb{Q} = \{x \in \mathbb{R} : x \in \mathbb{P}$ or $x \in \mathbb{Q}\}$.)

To cover this problem, one of the axioms in axiomatic set theory is there to guarantee the existence of the union of any two sets. .

Of course, this subtlety is easy to fix for intersections: we could have written out the definition more fully as : $A \cap B = \{x \in A : x \in B\}$

Here are few simple properties of unions and intersection. They're probably already familiar to you. Pay attention, though, to how the proofs are done. How do we prove that two sets are equal?

Theorem 1) $A \cup B = B \cup A$, and $A \cap B = B \cap A$
(commutative law for unions and intersections)

2) $A \cup (B \cup C) = (A \cup B) \cup C$, and $A \cap (B \cap C) = (A \cap B) \cap C$
(associative law for unions and intersections)

3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(distributive laws for unions and intersections)

Proof To prove two sets are equal we must show that they have the same elements. We do that by showing that if x is in the set on the left hand side (LHS) of the proposed equation, then x is also in the set on the right hand side (RHS) (thereby proving LHS $\subseteq$ RHS) and vice-versa. All parts of the theorem are very simple to prove. We illustrate by proving the last equality $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$:

If $x \in$ LHS $= A \cup (B \cap C)$ then $x \in A$ or $x \in B \cap C$.

If $x \in A$, then $x \in A \cup B$ and $x \in$ A $\cup C$, so $x \in (A \cup B) \cap (A \cup C) =$ RHS
If $x \in B \cap C$, then $x \in B$ and $x \in C$. Therefore $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C) =$ RHS

If $x \in$ RHS $= (A \cup B) \cap (A \cup C)$, then $x \in A \cup B$ and $x \in A \cup C$
Since $x \in A \cup B$, then $x \in A$ or $x \in B$.

If $x \in A$, then $x \in A \cup (B \cap C) =$ LHS
If $x \notin A$, then $x \in B$ and (since $x \in A \cup C$) we also have $x \in C$.
Therefore $x \in B \cap C$, so $x \in A \cup (B \cap C) =$ LHS. •

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