Sums of Powers of Natural Numbers

We'll use the symbol S_k for the sum of the k^{th} powers of the first n natural numbers. In other words,

$$S_k = 1^k + 2^k + \ \dots\ + n^k.$$

Of course, this is a "formula" for S_k, but it doesn't help you compute $-$ it doesn't tell you how to find the exact value, say, of $S_3 = 1^3 + 2^3 + \dots + 15^3$. We'd like to get what's called a <u>closed formula</u> for S_k, that is, one without the annoying " ... " in it.

For $S_0 = 1^0 + 2^0 + \ \dots\ + n^0$, this is easy: since there are n terms, each equal to 1, so we get

$$S_0 = 1 + 1 + \dots + 1 \ = 1 \cdot n \ = n$$

For S_1, it's already harder. Here's a slick way of finding a closed formula for S_1:

Write down S_1 twice, in two different orders:

$$\begin{array}{rccccccccccc} S_1 = & 1 & + & 2 & + & 3 & + \dots & + & (n-1) & + & n, & \text{and} \\ S_1 = & n & + & (n-1) & + & (n-2) & + \dots & + & 2 & + & 1 & \text{and then add to get:} \\ \hline 2S_1 = & (n+1) & + & (n+1) & + & (n+1) & + \dots & + & (n+1) & + & (n+1). & \end{array}$$

Since there are n terms on the right, each equal to $(n+1)$, we get

$$2S_1 = n(n+1), \text{ so}$$

$$S_1 \ = \frac{n(n+1)}{2}$$

This is a "usable" closed formula: for example, $1 + 2 + 3 + \dots + 15 \ = \frac{15(16)}{2} = 120.$

The argument above shows how somebody might actually "discover" the formula. Of course, <u>given</u> the proposed formula, we can also prove it by induction, just as we did in the lecture.

There are lots of such formulas. They often come up in Calculus I when integrals are introduced: they are useful for evaluating integrals like $\int_0^1 x\,dx,\ \int_0^1 x^2\,dx, \dots$ directly from the definition of the integral (without using the Fundamental Theorem of Calculus. (*For example, see pp. 357-358 in our current calculus text: Stewart, "Calculus:Concepts and Contexts" (3rd ed.)*) Here's a list of some of those formulas. You should try proving one or more of them using induction.

$$S_0 = 1^0 + 2^0 + \ \dots\ + \ n^0 = n$$
$$S_1 = 1^1 + 2^1 + \dots \ + n^1 = \frac{n(n+1)}{2}$$
$$S_2 = 1^2 + 2^2 + \dots \ + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$S_3 = [\tfrac{n(n+1)}{2}]^2 \quad \text{(Curious observation: } S_3 = [S_1]^2\text{)}$$

Where do these formulas come from? Each one can be proved by induction if you are given the formula. But what, for example is S_4 ? Did somebody find the S_3 formula by looking at lots of values of n and then guessing a formula that would fit the answers?

There is a way to get a formula for each S_k <u>once you know the previous ones</u>. In other words, there's a method for finding an "inductive" ("recursive") formula for each S_k in terms of the previous formulas S_0, S_1, ... , S_{k-1}. Here's how it works (an idea I first read in George Polya's book, *Mathematical Discovery*):

We can see directly that $S_0 = n$. How can we use S_0 to find S_1?

For any positive integer j, we know that $(j+1)^2 - j^2 = 2j+1$. We write this down for each value $j = 1, 2, ..., n$

$$\begin{aligned} 2^2 - 1^2 &= 2(1)+1 \\ 3^2 - 2^2 &= 2(2)+1 \\ 4^2 - 3^2 &= 2(3)+1 \\ &\dots \\ (n+1)^2 - n^2 &= 2(n)+1 \end{aligned}$$

Adding up the columns on both sides (with lots of cancellations on the left-hand side) gives

$$(n+1)^2 - 1 = 2(1+2+...+n)+n$$
$$= 2S_1 + n \quad \text{so}$$
$$n^2+2n+1-1 = 2S_1+n \quad \text{so}$$
$$n^2+n = 2S_1 \quad \text{so}$$
$$\frac{n^2+n}{2} = \frac{n(n+1)}{2} = S_1.$$

So now we know formulas for S_0 and S_1. How can we get a formula for S_2 ? It's the same idea, but a little more algebra.

For any j we know that $(j+1)^3 - j^3 = 3j^2 + 3j + 1$. We write this down for each $j = 1, 2, ..., n$.

$$\begin{aligned} 2^3 - 1^3 &= 3(1^2)+3(1)+1 \\ 3^3 - 2^3 &= 3(2^2)+3(2)+1 \\ 4^3 - 3^3 &= 3(3^2)+3(3)+1 \\ &\dots \\ (n+1)^3 - n^3 &= 3(n^2)+3(n)+1. \end{aligned}$$

Adding up the columns on both sides gives

$$(n+1)^3 - 1 = 3(1^2+2^2+...+n^2) + 3(1+2+...+n) + (1+1+..+1)$$
$$= 3 \quad S_2 \quad + 3 \quad S_1 \quad + \quad S_0, \quad \text{so}$$
$$n^3+3n^2+3n+1-1 = 3S_2+3S_1+S_0.$$

Then we solve to get S_2.

$$n^3+3n^2+3n-3S_1-S_0 = 3S_2, \text{ so}$$

$$S_2 = \frac{n^3+3n^2+3n-3S_1-S_0}{3}.$$ This is a recursive formula for S_2.

If we like, since we know formulas for S_1 and S_0, we can substitute, simplify, and get

$$S_2 = \frac{n^3+3n^2+3n-3[\frac{n(n+1)}{2}]-n}{3} =$$

$$= \frac{\frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2}}{3}$$

$$= \frac{2n^3+3n^2+n}{6} = \ldots = \frac{n(n+1)(2n+1)}{6}$$

See if you can now derive the formula for S_3 (given above), making use of the algebraic identity

$$(j+1)^4 - j^4 = 4j^3 + 6j^2 + 4j + 1.$$

Optional Material

If you know the binomial formula (from high school) and can therefore expand $(j+1)^k$, then the same idea works for any natural number k. But the bigger k is, the more lagebra is involved. An outline goes like this.

The formula for the "binomial coefficients" : $\binom{k}{l} = \frac{k!}{l!\,(k-l)!}$)

Suppose we have figured out formulas for $S_0, S_1, S_2, \ldots, S_{k-1}$. We know (from the binomial theorem) that for any j,

$$(j+1)^{k+1} - j^{k+1} = \binom{k+1}{1} j^k + \binom{k+1}{2} j^{k-1} + \binom{k+1}{3} j^{k-2} + \ldots + 1$$

Write this out for each value $j = 1, 2, \ldots n$.

$$\begin{array}{lll}
2^{k+1} & -1^{k+1} & = \binom{k+1}{1}1^k + \binom{k+1}{2}1^{k-1} + \binom{k+1}{3}1^{k-2} \ldots + 1 \\
3^{k+1} & -2^{k+1} & = \binom{k+1}{1}2^k + \binom{k+1}{2}2^{k-1} + \binom{k+1}{3}2^{k-2} \ldots + 1 \\
 & \ldots\ldots & \\
(n+1)^{k+1} & -n^{k+1} & = \binom{k+1}{1}n^k + \binom{k+1}{2}n^{k-1} + \binom{k+1}{3}n^{k-2} \ldots + 1.
\end{array}$$

<u>Add the columns:</u>

$$(n+1)^{k+1} - 1 = \binom{k+1}{1}(1^k + 2^k + \ldots + n^k) + \binom{k+1}{2}(1^{k-1} + 2^{k-1} + \ldots + n^{k-1})$$

$$+ \binom{k+1}{3}(1^{k-2} + 2^{k-2} + \ldots + n^{k-1}) \quad \ldots + n$$

$$= \binom{k+1}{1} S_k + \binom{k+1}{2} S_{k-1} + \binom{k+1}{3} S_{k-2} + \ \ldots \ + S_0.$$

Then we solve for what we want:

$$\begin{aligned}
S_k &= [(n+1)^{k+1} - 1 - \tbinom{k+1}{2} S_{k-1} - \tbinom{k+1}{3} S_{k-2} - \ \ldots \ - S_0] \,/\, \tbinom{k+1}{1} \\
&= [(n+1)^{k+1} - 1 - \tbinom{k+1}{2} S_{k-1} - \tbinom{k+1}{3} S_{k-2} - \ \ldots \ - S_0] \,/\, (k+1)
\end{aligned}$$

We are assuming that we already have formulas for $S_{k-1}, S_{k-2}, \ldots S_1, S_0$ $-$ which we then substitute into this formula to get one closed, if complicated, formula for S_k in terms of n. Try it to find a formula for

$$S_4 = 1^4 + 2^4 + \ldots + n^4 = \ldots$$