Math 417, Fall 2009
Exam 2
Take Home: due in class on Tuesday, November 10

Do all three problems in Part I, one from Part II and one from Part III. Write up your solutions in the same way you would write up homework problems solutions.

You can cite any theorems, exercises or examples in the textbook if they were proved in the notes or assigned as homework. But if you cite a theorem left unproved in the text, or an exercise not assigned, then you need to provide a proof.

No references other than the textbook and class notes can be used. No discussion (real or electronic) about any part of the exam with any other person is allowed until after all exam solutions are handed in.

Part I Do all three problems 1)-3)

1. a) Let $X = \{a, b, c, d\}$, $\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$
Find the smallest topology $\mathcal{T}$ that contains both $\mathcal{T}_1$ and $\mathcal{T}_2$.

b) Consider the space $(\mathbb{R}, \mathcal{T})$ where $\mathcal{T} = \{\emptyset\} \cup \{O \subseteq \mathbb{R} : \{-1, 1\} \subseteq O\}$.

i) What is the simplest possible neighborhood base $\mathcal{B}_x$ at a point $x \in \mathbb{R}$?

ii) Explain why $(\mathbb{R}, \mathcal{T})$ is or is not Lindelöf.

iii) In $(\mathbb{R}, \mathcal{T})$: what are int $\mathbb{Q}$, int $\mathbb{P}$, cl $\{2, 4, 6, ...\}$ and cl$\{1, 3, 5, ...\}$?

iv) Describe the sequences (x_n) in $(\mathbb{R}, \mathcal{T})$ for which $(x_n) \to \frac{1}{2}$.

v) Let $\mathcal{U}$ be the usual topology on $\mathbb{R}$. Give an example of a nonconstant continuous function $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{U})$, or prove that no such function exists.

c) Give an example of <u>nonempty</u> spaces X, Y, Z such that $X \times Y$ is homeomorphic to $X \times Z$ but Y is not homeomorphic to Z.

2. You may also assume that each integer (except $1, -1$) is divisible by a prime but nothing else about prime numbers. (*This assumption is equivalent to assuming that each natural number bigger than* 1 *can be factored into primes*).

For $a \in \mathbb{Z}$ and $d \in \mathbb{N}$, let

$$B_{a,d} = \{..., a-2d, a-d, a, a+d, a+2d, ...\} = \{a + kd : k \in \mathbb{Z}\}$$

and let $\mathcal{B} = \{B_{a,d} : a \in \mathbb{Z}, d \in \mathbb{N}\}$ (*so* $\mathcal{B}$ *is the set of all arithmetic progressions in* $\mathbb{Z}$)

a) Prove that $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on $\mathbb{Z}$.

b) Show that each set $B_{a,d}$ is closed in $(\mathbb{Z}, \mathcal{T})$.

c) What is the set $\bigcup\{B_{0,p} : p$ a prime number$\}$? Explain why this set is not closed in $(\mathbb{Z}, \mathcal{T})$.

d) What does part c) tell you about the set of prime numbers?

3. In any space $(X, \mathcal{T})$ we can define a relation $\preceq$ as follows:

$$\text{for } x, y \in X,\ x \preceq y \text{ iff } x \in \text{cl}\,\{y\}.$$

If we write the relation as a set of pairs, then

$$\preceq\ = \{(x, y) \in X \times X : x \in \text{cl}\{y\}\}$$

a) In $\mathbb{R}$ (with the usual topology), then the set $\preceq\ =$???

b) If $\mathcal{T}$ is the right ray topology on $\mathbb{R}$, then the set $\preceq\ =$???

(***) <u>For the remaining parts of this problem</u>, suppose the space $(X, \mathcal{T})$ has the following property:

if $x \neq y \in X$, then $\mathcal{N}_x \neq \mathcal{N}_y$
(*that is, different points have different neighborhood systems*).

For example, every T_1-space has this property.

c) In $(X, \mathcal{T})$:

i) $x \preceq x$ for all $x \in X$ (*this is obvious; don't bother with the one line proof)*

Prove that

ii) for all $x, y \in X$: if $x \preceq y$ and $y \preceq x$, then $x = y$

iii) for all $x, y, z \in X$: if $x \preceq y$ and $y \preceq z$, then $x \preceq z$

A relation that satisfies i), ii) and iii) is called a <u>partial order</u>; $\preceq$ is a partial ordering in any topological space that satisfies condition (***).

d) Suppose $x_1, ..., x_k$ are distinct points in X and that $x_1 \preceq x_2 \preceq ... \preceq x_k$. Prove that x_k is isolated in $A = \{x_1, ..., x_k\}$.

e) Prove that if A is any finite subset of X, then some point of A is isolated in A. (*Caution: see part a)*)

Part II Do <u>one</u> of the following problems: 4) or 5)

4. a) Suppose f and g are continuous functions from a space X into a Hausdorff space Y. Prove that $C = \{x \in X : f(x) = g(x)\}$ is closed in X.

b) Suppose $f : X \to Y$ is onto and open (but not necessarily continuous), and suppose that the set $F = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is closed in $X \times X$. Prove that Y is Hausdorff.

5. Suppose X is a Hausdorff space (= T_2-space). A subspace $A \subseteq X$ is called a <u>retract</u> of X if there exists a continuous function $f : X \to A$ such that $f(a) = a$ for all $a \in A$. The function r is called a retraction.

<u>Example</u>: $f : (\mathbb{R}^2 - \{0\}) \to S^1$ given by $p \to p/||\,p\,||$ is a retraction of the "punctured plane" onto the unit circle.

a) Prove that if A is a retract of X, then A is closed in X.

b) Let O be a nonempty open set in X, where X has the cofinite topology. Prove that O is a retract of X (so that part a) might not be true if "Hausdorff" is omitted from the hypothesis).

Part III Do <u>one</u> of the following two problems: 6) or 7)

6. Let X be a Hausdorff space with a dense subset D, and suppose f is a homeomorphism from D onto a topological space Y. Prove that f cannot be continuously extended to any point $a \in X - D$. (*Stated more precisely, this means: show that if $a \in X - D$, then there cannot be a continuous map $g : D \cup \{a\} \to Y$ for which $g|D = f$.*)

7. Prove that if X is a Hausdorff space (= T_2-space) and D is a dense set, then $|X| \leq 2^{2^{|D|}}$

(*Hint: if $x \neq y$, then there are open sets for which $x \in U$, $y \in V$ and $U \cap D \neq V \cap D$.*

Caution: sequences may not be sufficient to describe closures in X.)