Math 417, Fall 2009
Some Problems on Connectedness
You do not need to hand in these problems.

1. Let $(X, d)$ and $(Y, s)$ be two unbounded connected metric spaces. Give $X \times Y$ the product topology. For $(a, b) \in X \times Y$ and $k>0$, let $K=\{(x, y) \in X \times Y: d(x, a) \leq k$ and $s(y, b) \leq k\}$. Prove that the complement of $K$ in $X \times Y$ is connected.
2. Suppose $(X, d)$ is a connected metric space with $|X|>1$. Prove that $|X| \geq c$.
3. A metric space $(X, d)$ satisfies the $\epsilon$-chain condition if, for all $x, y \in X$ and for all $\epsilon>0$, there exists a finite set of points $x=x_{1}, x_{2}, \ldots, x_{n}=y$ such that for all $i=1, \ldots, n-1, d\left(x_{i}, x_{i+1}\right)<\epsilon$.
a) Prove that if $(X, d)$ is connected, then $(X, d)$ satisfies the $\epsilon$-chain condition. (Hint: Let $x \in X$ and $\epsilon>0$, consider the set of all points $y$ that can be "chained" to $x$. Is this set open? ... )
b) Give an example of a metric space $(X, d)$ which satisfies the $\epsilon$-chain condition but which is not connected.
c) Prove that if $(X, d)$ is a compact metric space that satisfies the $\epsilon$-chain condition, then $(X, d)$ is connected.
4. Prove or disprove: there is a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f[\mathbb{P}] \subseteq \mathbb{Q}$ and $f[\mathbb{Q}] \subseteq \mathbb{P}$. (Hint: what you can say about the range of $f$.)
5. a) Find the cardinality of the set of all compact connected subsets of the plane.
b) Find the cardinality of the set of all connected subsets of the plane.
