Math 417, Fall 2009
Some Problems on Connectedness

You do not need to hand in these problems.

1. Let $(X, d)$ and $(Y, s)$ be two unbounded connected metric spaces. Give $X \times Y$ the product topology. For $(a, b) \in X \times Y$ and $k > 0$, let $K = \{(x, y) \in X \times Y: d(x, a) \leq k$ and $s(y, b) \leq k\}$. Prove that the complement of $K$ in $X \times Y$ is connected.

2. Suppose $(X, d)$ is a connected metric space with $|X| > 1$. Prove that $|X| \geq c$.

3. A metric space $(X, d)$ satisfies the $\epsilon$-chain condition if, for all $x, y \in X$ and for all $\epsilon > 0$, there exists a finite set of points $x = x_1, x_2, \ldots, x_n = y$ such that for all $i = 1, \ldots, n - 1, d(x_i, x_{i+1}) < \epsilon$.

   a) Prove that if $(X, d)$ is connected, then $(X, d)$ satisfies the $\epsilon$-chain condition. (Hint: Let $x \in X$ and $\epsilon > 0$, consider the set of all points $y$ that can be "chained" to $x$. Is this set open? ...)

   b) Give an example of a metric space $(X, d)$ which satisfies the $\epsilon$-chain condition but which is not connected.

   c) Prove that if $(X, d)$ is a compact metric space that satisfies the $\epsilon$-chain condition, then $(X, d)$ is connected.

4. Prove or disprove: there is a continuous function $f: \mathbb{R} \to \mathbb{R}$ such that $f[\mathbb{P}] \subseteq \mathbb{Q}$ and $f[\mathbb{Q}] \subseteq \mathbb{P}$.
   (Hint: what you can say about the range of $f$.)

5. a) Find the cardinality of the set of all compact connected subsets of the plane.

   b) Find the cardinality of the set of all connected subsets of the plane.