## Math 417, Fall 2009

## Homework 1

Homework 1 will be due in class on Thursday, September 10. Some of the problems listed here will require material from the lectures on September 8, and maybe a bit of September 10, but several you can work on immediately. After homework is are turned in, I will post solutions (sometimes complete, sometimes partial) online the syllabus.

From the syllabus:
"Students are encouraged to discuss homework assignments with each other; you should share questions and ideas. It is a powerful way to learn the concepts. Each student, however, must write up the homework solutions independently in his/her own words and notation. One handy way to avoid "borrowing too much" from sessions with others is to talk together but not take any written notes away from the conversation. Suspicious similarities between solution sets may be noted by the grader and may result in a grade of 0 for the homework. "

Please do your work first on scrap paper. After you've figured out what you want to do, then write up a final, polished version of your solutions on $81 / 2 \times 11$ paper with smooth edges (no ragged pages torn from a spiral notebook).

Do not hand in problems A) or B) but be sure you understand the notation:
A) Which of the following are true when " $\in$ " is inserted in the blank space? Which are true when " $\subseteq$ " is inserted?
a) $\{\emptyset\} \ldots\{\emptyset,\{\emptyset\}\}$
b) $\{\emptyset\} \quad \_\{\emptyset,\{\{\emptyset\}\}\}$
c) $\{\{\emptyset\}\}_{\_}\{\emptyset,\{\emptyset\}\}$
d) $\{\{\emptyset\}\}_{—}\{\emptyset,\{\{\emptyset\}\}\}$
e) $\{\{\emptyset\}\}_{\_}\{\emptyset,\{\emptyset,\{\emptyset\}\}\}$
B) Prove or give a counterexample:
a) if $A$ and $B$ are equivalent and $C$ and $D$ are equivalent, then $A \cap C \sim B \cap D$
b) if $A \sim B$, then $A-B \sim B-A$
c) if $A-B \sim B-A$, then $A \sim B$
d) if $A, B$, and $C$ are nonempty sets and $A \times B \sim A \times C$, then $B \sim C$

## To hand in:

1. For any sets $A$ and $B$ it is true that $\mathcal{P}(A) \cap \mathcal{P}(B)=\mathcal{P}(A \cap B)$ and that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. State and prove a theorem of the form:

$$
\mathcal{P}(A) \cup \mathcal{P}(B)=\mathcal{P}(A \cup B) \text { if and only if ... }
$$

2. Suppose $A, B, C$, and $D$ are sets, with $A \neq \emptyset$ and $B \neq \emptyset$. Show that if

$$
(A \times B) \cup(B \times A)=(C \times D) \cup(D \times C),
$$

then either $(A=C$ and $B=D)$ or $(A=D$ and $B=C)$.
3. Let $\mathbb{R}$ denote the real numbers and $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x)=\left|3 x^{2}-2 x+1\right|$ and $g(x)=2 x-1$. Find the range of $f \circ g$.
4. Let $\mathcal{L}$ be the set of straight lines in $\mathbb{R}^{2}$ which do not pass through the origin ( 0,0 ). Describe geometrically a function $f: \mathcal{L} \rightarrow \mathbb{R}^{2}-\{(0,0)\}$ where $f$ is a bijection. (By "geometrically" $I$ mean: give an answer of the form "for any straight line $\ell$, define $f(\ell)$ to be the point $(x, y) \in \mathbb{R}^{2}-\{(0,0)\}$ which we obtain by ...").
5. Let $f: X \rightarrow X$ and let $f^{n}: X \rightarrow X$ denote the result of composing $f$ with itself $n$ times. Suppose that for every $x \in X$, there exists an $n \in \mathbb{N}$ such that $f^{n}(x)=x$ (note that $n$ may depend on $x$ ). Prove that $f$ is a bijection.
6. Let $C(\mathbb{R})$ denote the set of all continuous real-valued functions with domain $\mathbb{R}$, that is, $C(\mathbb{R})=\left\{f: f \in \mathbb{R}^{\mathbb{R}}\right.$ and $f$ is continuous $\}$. Define a map $I: C(\mathbb{R}) \rightarrow C(\mathbb{R})$ as follows:
for $f \in C(\mathbb{R}), I(f)$ is the function given by $I(f)(x)=\int_{0}^{x} f(t) d t$..
Is I one-to-one? onto? (Hint: The Fundamental Theorem of Calculus is useful here.)
7. Let $A_{1}, A_{2}, \ldots, A_{n}, \ldots$ be subsets of a set $A$. Define $\underline{\lim } A_{n}=\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_{k}$.
( lim is read "lim inf". In expanded form, the notation means that

$$
\left.\underline{\lim } A_{n}=\left(A_{1} \bigcap A_{2} \bigcap A_{3} \bigcap \ldots\right) \cup\left(A_{2} \bigcap A_{3} \bigcap A_{4} \bigcap \ldots\right) \cup\left(A_{3} \bigcap A_{4} \bigcap A_{5} \bigcap \ldots\right) \cup \ldots\right)
$$

Similarly, define $\overline{\lim } A_{n}=\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_{k}$. ( $\overline{\lim }$ is read "lim sup")
a) Prove that $\underline{\lim } A_{n}=\left\{x: x\right.$ is in all but at most finitely many $\left.A_{n}\right\}$ and that $\overline{\lim } A_{n}$ $=\left\{x: x\right.$ is in infinitely many of the $\left.A_{n}\right\}$.
b) Prove that $\bigcap_{n=1}^{\infty} A_{n} \subseteq \underline{\lim } A_{n} \subseteq \overline{\lim } A_{n} \subseteq \bigcup_{n=1}^{\infty} A_{n}$.
(You might also want to think about parts c), d) of Problem E9 in the text.)
8. Let $\mathcal{B}$ be the collection of infinite subsets of $\mathbb{N}$. Define $f: \mathcal{B} \rightarrow(0,1]$ as follows:

For $B \in \mathcal{B}, f(B)=$ "binary decimal" $0 . x_{1} x_{2} x_{3} \ldots x_{n} \ldots$ where $x_{n}=1$ if $n \in B$ and $x_{n}=0$ if $n \notin B$.
(For example, if $\mathbb{E}=\{2,4,6, \ldots\} \in \mathcal{B}$, then $f(\mathbb{E})=0.010101 \ldots$..base 2 )
Prove or disprove that $f$ is onto.

