Math 417, Fall 2009
Homework 2

Homework 1 will be due in class on Tuesday, September 22.

Problems to think about (not to hand in):

A. True or false:

a) if A is infinite and B is countable, then $A \cup B \sim B$
b) if A is infinite and B is countable, then $A \cup B \sim A$

B. What is wrong with the following argument?

For each irrational number p, pick an open interval (a_p, b_p) with rational endpoints and centered at p. Since $\mathbb{Q} \times \mathbb{Q}$ is countable, there are only countably many possible pairs (a_p, b_p). Furthermore, the mapping $\Phi: \mathbb{P} \to \mathbb{Q} \times \mathbb{Q}$ given by $\Phi(p) = (a_p, b_p)$ is one-to one, since (a_p, b_p) is centered at p. Therefore $\mathbb{P}$ is equivalent to a subset of $\mathbb{Q} \times \mathbb{Q}$, so $\mathbb{P}$ is countable.

C. Prove or give a counterexample for the following statement:

If $\mathcal{C}$ is an uncountable collection of uncountable subsets of $\mathbb{R}$, then at least two sets in $\mathcal{C}$ must have uncountable intersection.

To hand in:

1. If A is uncountable and B is countable, prove that $A \sim A - B$.

2. A subset B of X is called *cocountable* if $X - B$ is countable and *cofinite* if $X - B$ is finite. (*The name is an abbreviation: cocountable $=$ complement is countable.*)

a) Prove that if B and C are cocountable subsets of X, then $B \cup C$ and $B \cap C$ are also cocountable. Is the analogous result true for cofinite sets?

b) Show how to write the set of irrational numbers, $\mathbb{P}$, as an intersection of countably many cofinite subsets of $\mathbb{R}$.

3. A collection $\mathcal{A}$ of sets is called pairwise disjoint if whenever $A, B \in \mathcal{A}$ and $A \neq B$, then $A \cap B = \emptyset$. For each statement, provide a proof or a counterexample:

a) If $\mathcal{A}$ is a collection of pairwise disjoint circles in the plane, then $\mathcal{A}$ is countable.

b) If $\mathcal{A}$ is a collection of pairwise disjoint circular disks in the plane, then $\mathcal{A}$ is countable.

4. A sequence s in $\mathbb{N}$ is called eventually constant if $\exists k, l \in \mathbb{N}$ such that $s_n = l$ for all $n \geq k$. Prove that the set of eventually constant sequences in $\mathbb{N}$ is countable.

5. Let A be an uncountable subset of $\mathbb{R}$. Prove that there is a subset of distinct elements $\{a_n : n = 1, 2, ...\} \subseteq A$ such that $\sum_{n=1}^{\infty} a_n$ diverges.

6. Suppose S is a countable subset of $\mathbb{R}$. Prove that there exists a fixed real number, c such that $s + c$ is transcendental for every $s \in$ S. (*Hint: if $s \in S$, then for how many values of r can $s + r$ be algebraic?*)

7. a) Show how to write $\mathbb{N}$ as the union of infinitely many pairwise disjoint infinite subsets.

b) Show how to write $\mathbb{N}$ as the union of uncountably many sets with the property that any two of them have finite intersection. (*Such sets are called almost disjoint.*)
(*Hint: This is a statement that actually is true for any infinite countable set, not just for $\mathbb{N}$. For part b), you may find it easier to solve the problem for $\mathbb{Q}$ instead, and then use a bijection to"convert" your solution for $\mathbb{Q}$ into a solution for the set $\mathbb{N}$.*)

8. Let *D* be a countable set of points in the plane, $\mathbb{R}^2$. Prove there exist sets A and B such that
$D = A \cup B$, where the set A has finite intersection with every horizontal line in the plane and B has finite intersection with every vertical line in the plane.

Notes: 1) This problem is fairly hard. You might get an idea by starting the easy special case of $D = \mathbb{N} \times \mathbb{N}$ 2) The statement that "$\mathbb{R}^2$ can be written as the union of two sets A and B where A has countable intersection with every horizontal line and B has countable intersection with every vertical line" is, in fact, equivalent to the continuum hypothesis (*see p. 40*) *!*