Math 417, Fall 2009
Homework 3

Homework 3 will be due in class on Thursday, October 1.

Problems to think about (not to hand in):

A) $\mathbb{R} = \mathbb{Q} \cup \mathbb{P}$. Since $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable, $\mathbb{P}$ must be uncountable. But that does not mean (unless you assume CH) that $|\mathbb{P}| = c$. Explain why $|\mathbb{P}| = c$ using Example 14.5(4) from Chapter 1.

B) Find all unjustified steps in the following "proof" of the continuum hypothesis:

> *If CH is false, then $\aleph_0 < m < c$ for some cardinal m. Since $c = \aleph_0^{\aleph_0} \le m^{\aleph_0} \le c^{\aleph_0} = c$, we have $m^{\aleph_0} = c = 2^{\aleph_0} < 2^m$, so $m^{\aleph_0} < 2^m$. Therefore $(m^{\aleph_0})^c < (2^m)^c$, so $m^c < 2^c$, which is impossible because $m > 2$.. Therefore no such m can exist, so CH is true.*

C) The following statements refer to a metric space (X, d). Prove the true statements and give counterexamples for the false ones. (*The statements illustrate the danger of assuming that familiar features of $\mathbb{R}^n$ necessarily carry over to arbitrary pseudometric spaces.*)

a) $B_\epsilon(x) = B_\epsilon(y)$ implies $x = y$ (i.e., "a ball can't have two centers")

b) The diameter of a set A in a metric space (X, d) is defined by:

$$\text{diam}(A) = \{d(x, y) : x, y \in A\} \le \infty.$$

The diameter of $B_\epsilon(x)$ must be bigger than ϵ.

c) $B_\epsilon(x)$ is never a closed set.

D) a) Give an example of a metric space (X, d) with a proper nonempty clopen subset.
b) Give an example of a metric space (X, d) and a subset that's neither open nor closed.
c) Give an example of a metric space (X, d) and a subset which contains no point that is not a limit point of the set. (*Note: a point x is a limit point of a set A if, for every open set O containing x, $O \cap (A - \{x\}) \neq \emptyset$.*)

To hand in:

1. Find the cardinal number of each set:

a) the set of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$
b) the set of all strictly increasing sequences $f : \mathbb{N} \to \mathbb{N}$ (*Note: f is <u>strictly</u> <u>increasing</u> if $l, m \in \mathbb{N}$ and $l < m \Rightarrow f(l) < f(m)$.*)

Note: Sometimes one proves $|A| = m$ by giving two separate arguments, one to show $|A| \leq m$ and the other to show $|A| \geq m$. One of these two inequalities often is easy and all the harder job is to show that the other inequality holds.

2. Prove that for every infinite set E, there is an infinite sequence of pairwise disjoint subsets $E_1, E_2, E_3, \ldots$ such that $E = \bigcup_{n=1}^{\infty} E_n$ and $|E_n| = |E|$ for all n. (*Hint: We stated a multiplication rule that $m \cdot \aleph_0 = m$ for any infinite cardinal m. You can assume that rule.*)

3. Suppose (X, d) is a metric space. Define $d^*(x, y) = \min\{1, d(x, y)\}$. Prove that d^* is also a metric on X, and that $\mathcal{T}_d = \mathcal{T}_{d^*}$.

(*Since $d^*(x, y) \leq 1$ for all $x, y \in X$, we say that the metric d^* is a <u>bounded</u> metric. The exercise shows that for any metric d there is an equivalent bounded metric – one that produces the same open sets. "Boundedness" is a property determined by the metric, <u>not</u> by the topology.*)

4. Suppose (X, d) is a metric space in which every intersection of open sets is open. Prove that every subset of X is open.

5. Suppose that O is a finite open set in a metric space (X, d). Prove that every point x in O is an isolated point in (X, d).

6. We know that in general an infinite union of closed sets may not be closed. However, an infinite union of closed sets may sometimes be closed if the closed sets are "spread out enough" from each other. For example,

a) Suppose that for each $n \in \mathbb{N}$, F_n is a closed set in $\mathbb{R}$ and that $F_n \subseteq (n, n+1)$. Prove that $\bigcup_{n=1}^{\infty} F_n$ is closed in $\mathbb{R}$.

b) More generally: suppose for $\alpha \in A$, each F_α is a closed set in (X, d) and that for each point $x \in X$ there is an $\epsilon > 0$ such that $B_\epsilon(x)$ has nonempty intersection with at most finitely many F_α's. Prove that $\bigcup_{\alpha \in A} F_\alpha$ is closed in (X, d).

7. Let p be a fixed prime number. We define the <u>p-adic absolute value</u> $| \ |_p$ (sometimes called the p-adic norm) on the set of rational numbers $\mathbb{Q}$ as follows:

> If $0 \neq x \in \mathbb{Q}$, write $x = \frac{p^k m}{n}$ for integers k,m,n, where p does not divide m or n, and define $|x|_p = p^{-k} = \frac{1}{p^k}$. (*Of course, k may be negative.*). Also, define $|0|_p = 0$.

Prove that $| \ |_p$ "behaves the way an absolute value (norm) should" $-$ that is, for all $x, y \in \mathbb{Q}$

a) $|x|_p \geq 0$ and $|x|_p = 0$ iff $x = 0$
b) $|xy|_p = |x|_p \cdot |y|_p$
c) $|x + y|_p \leq |x|_p + |y|_p$

$| \ |_p$ actually satisfies a <u>stronger</u> inequality than the inequality in part c). Prove that

d) $|x + y|_p \leq \max\{|x|_p, |y|_p\} \leq x|_p + |y|_p$

Whenever we have an absolute value (norm), we can use it to define a distance function: :

for $x, y \in \mathbb{Q}$, let $d_p(x, y) = |x - y|_p$

e) Prove that d_p is a metric on $\mathbb{Q}$ and show that d_p in fact satisfies an inequality <u>stronger</u> than the usual triangle inequality, namely:

for all $x, y, z \in \mathbb{Q}$, $d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z\})$.

f) Give a specific example for x, y, z, p for which

$d_p(x, z) < \max\{d_p(x, y), d_p(y, z\})$

(*Hint: It's convenient to be able to refer to the exponent "k" associated with a particular x. If $x = \frac{p^k m}{n}$, then k roughly refers to the "number of p's that can be factored out of x" $-$ so we can call $k = \nu(x)$.*

Prove that $\nu(a - b) \geq \min\{\nu(a), \nu(b)\}$ whenever $a,b \in \mathbb{Q}$ with $a,b \neq 0$ and $a \neq b$. Note that strict inequality can occur here: for example, when $p = 3$, $\nu(8) = \nu(2) = 0$, but $\nu(8 - 2) = \nu(6) = 1$.)