Math 417, Fall 2009
Homework 4

Homework 4 will be <u>due in class on Thursday, October 15</u>.

1. Suppose (X,d) is a metric space and $x \in X$. Prove that the following two statements are <u>equivalent</u>:

i) x is not an isolated point of X
ii) every open set containing x contains an infinite number of points.

2. The <u>Hilbert cube</u>, *H*, is a certain subset of ℓ_2: $H = \{x \in \ell_2 : \ |x_i| \le \frac{1}{i}\}$. Prove that *H* is closed in ℓ_2. (*Not to hand in: is H also open in* ℓ_2 ? *Prove or disprove.*)

3. A subset A of a space (X,d) is called a G_δ set if A can be written as a countable intersection of open sets; A is called an F_σ set if A can be written as a countable union of closed sets.

The names G_δ and F_σ go back to the classic book <u>Mengenlehre</u> of the German mathematician Felix Hausdorff. The "G" and the "F" represent "open" and "closed"; σ and the δ in the notation represent abbreviations for the German words used for union and intersection: Summe and Durchschnitt.

a) Prove that in a pseudometric space (X,d) every closed set is a G_δ set and every open set is an F_σ set.

b) In $\mathbb{R}$, the set $\mathbb{P}$ is a G_δ because we can write $\mathbb{P} = \bigcap_{q\in\mathbb{Q}} O_q$, where O_q is the open set $\mathbb{R} = \{q\}$. In Chapter 4, however, we will prove that $\mathbb{Q}$ is <u>not</u> a G_δ set in $\mathbb{R}$.
<u>Find the error</u> in the following argument which "proves" that <u>every</u> subset of $\mathbb{R}$ is a G_δ set:

Let $A \subseteq \mathbb{R}$. For $x \in A$, let $J_n = \bigcup\{B_{\frac{1}{n}}(x) : x \in A\}$. J_n is open for each $n \in \mathbb{N}$. Since $\{x\} = \bigcap_{n=1}^{\infty} B_{\frac{1}{n}}(x)$, it follows that $A = \bigcap_{n=1}^{\infty} J_n$, so A is a countable intersection of open sets, that is, A is a G_δ set.

4. Let (X,d) be a pseudometric space. Suppose that for every $\epsilon > 0$, there exists a countable subset D_ϵ of *X* with the following property: $\forall x \in X,\ \exists y \in D_\epsilon$ such that $d(x,y) < \epsilon$. Prove that (X,d) is separable.

5. a) Suppose A is a closed set in the pseudometric space (X,d) and that $x_0 \notin A$. Prove that there is a continuous function $f : X \to [0,1]$ such that $f|A = 0$ and $f(x_0) = 1$. (*Hint: Consider the function "distance to the set A."*)

b) Suppose A and B are disjoint closed sets in (X,d). Prove that there exists a continuous function $f : X \to \mathbb{R}$ such that $f|A = 0$ and $f|B = 1$. (*Hint: Consider* $\frac{d(x,A)}{d(x,A)+d(x,B)}$)

c) Using b) (*or by another method*) prove that is A and B are disjoint closed sets in (X, d), then there are open sets U and V for which $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

6. A function $f: (X, d) \to (Y, s)$ is called an <u>isometry between X and Y</u> if f is <u>onto</u> and, for all $x, y \in X$, $d(x, y) = s(f(x), f(y))$. If such an f exists, we say that (X, d) and (Y, s) are <u>isometric</u> to each other. If f is <u>not</u> onto, we say f is an isometry of X <u>into</u> Y, or that f is an <u>isometric embedding of (X, d) into (Y, s)</u>.

Let $\mathbb{R}$ and $\mathbb{R}^2$ have their usual metrics.

a) Prove that there is no isometry between $\mathbb{R}$ and $\mathbb{R}^2$.

b) Let $a \in \mathbb{R}$. Prove that there are exactly two isometries from $\mathbb{R}$ onto $\mathbb{R}$ which hold the point a fixed (that is, for which $f(a) = a$).

c) Give an example of a metric space which is isometric to a proper subset of itself.

7. (<u>The Pasting Lemma</u>) *The two parts of this problem give conditions when a collection of <u>continuous</u> functions defined on subsets of X can be "united" (= "pasted together") to form a new <u>continuous</u> function.* Let A be an indexing set, and suppose the sets O_α $(\alpha \in A)$ are open in (X, d) and that the sets F_α $(\alpha \in A)$ are closed in (X, d).

a) Suppose that functions $f_\alpha : O_\alpha \to (Y, s)$ are continuous and that, if $\alpha \neq \beta$, then $f_\alpha|(O_\alpha \cap O_\beta) = f_\beta|(O_\alpha \cap O_\beta)$ (*that is, f_α and f_β agree where their domains overlap*).
Then $\bigcup_{\alpha \in A} f_\alpha = f : \bigcup_{\alpha \in A} O_\alpha \to Y$ is continuous.

b) Suppose that for each $i = 1, \ldots, n$, $f_i : F_i \to (Y, s)$ is continuous and that, if $i \neq j$, then $f_i|(F_i \cap F_j) = f_j|(F_i \cap F_j)$ (*that is, f_i and f_j agree where their domains overlap*).
Then $f = \bigcup_{i=1}^n f_i : \bigcup_{i=1}^n F_i \to Y$ is a continuous function.

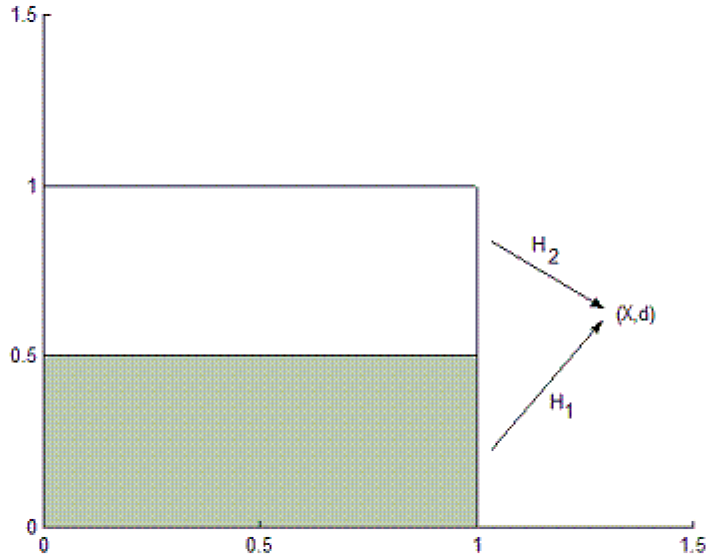
c) Suppose $\{F_\alpha : \alpha \in A\}$ has the property that each point $x \in X$ has a neighborhood N_x such that N_x has nonempty intersection with only finitely many of the F_α's. (*A family if sets $\{F_\alpha : \alpha \in A\}$ with this property is called <u>locally finite</u>.*) Then $\bigcup_{\alpha \in A} F_\alpha$ is closed. (*You proved this in Problem 6b) of Homework 3 – although the wording there involved ϵ-balls rather than neighborhoods. <u>Don't</u> prove this again!*)

Suppose that we have continuous functions $f_\alpha : F_\alpha \to (Y, s)$ and that, if $\alpha \neq \beta \in A$, then $f_\alpha|(F_\alpha \cap F_\beta) = f_\beta|(F_\alpha \cap F_\beta)$ (*that is, f_α and f_β agree where their domains overlap*).
Let $f = \bigcup_{\alpha \in A} f_\alpha$. Then the function $f : \bigcup_{\alpha \in A} F_\alpha \to Y$ is continuous.

Note: the most common use of the Pasting Lemma is when the index set A is finite (in which case $\{F_\alpha : \alpha \in A\}$ is certainly locally finite!). See the following page.

For example, suppose

$H_1 : [0,1] \times [0, \frac{1}{2}] \to (X,d)$ *is continuous, and*
$H_2 : [0,1] \times [\frac{1}{2}, 1] \to (X,d)$ *is continuous, and*
$H_1(t, \frac{1}{2}) = H_2(t, \frac{1}{2})$ *for all* $t \in [0,1]$



H_1 is defined on the lower closed half of the box $[0,1]^2$, H_2 is defined on the upper closed half, and they agreed on the "overlap" – that is, on the horizontal line segment $[0,1] \times \{\frac{1}{2}\}$. Part c) says the two functions can be pieced together into a <u>continuous function</u> $H : [0,1]^2 \to (X,d)$, where $H = H_1 \cup H_2$.