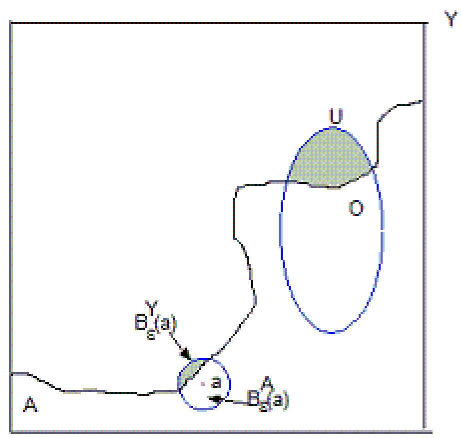
For Problem 7 in Homework 4, you might find the following simple observations helpful:

If (Y, d) is any pseudometric space and $A \subseteq Y$, then we can also think of (A, d) as a pseudometric space $-$ that is, we use the same d to measure distances between points in A.

Then we can talk about open and closed sets in the space (A, d), as well as open and closed sets in the larger space (Y, d). How are these related?

Notice that if $a \in A$, then $\{x \in A : d(x, a) < \epsilon\} = A \cap \{x \in Y : d(x, a) < \epsilon\}$ so that (the ϵ-ball in (A, d) centered at $a \in A$) is the same as $A \cap$ (ϵ-ball in (Y, d) centered at a):

for $a \in A$, $B_\epsilon^A(a) = A \cap B_\epsilon^Y(a)$ (*)



With this observation about balls it is easy to describe how open sets in A and in Y are related:

Theorem Suppose $A \subseteq (Y, d)$ and that $O \subseteq A$. Then O is an open set in (A, d) iff there is an open set U in (Y, d) for which $O = A \cap U$.

Proof $\Rightarrow$ If O is open in (A, d), then O is a union of balls in A. So using the observation (*), we can write

$$O = \bigcup_{a \in A} B_{\epsilon_a}^A(a) = \bigcup_{a \in A}(A \cap B_{\epsilon_a}^Y(a)) = A \cap (\bigcup_{a \in A} B_{\epsilon_a}^Y(a))$$
$$= A \cap U \text{ where } U \text{ is the open set } \bigcup_{a \in A} B_{\epsilon_a}^Y(a) \text{ in } Y.$$

$\Leftarrow$ Suppose $O = A \cap U$, where U is an open set in (Y, d). We want to show O is open in (A, d). Let $a \in O$. Then $a \in U$ and U is open in (Y, d) so for some $\epsilon > 0$, $B_\epsilon^Y(a) \subseteq U$. But then, by (*), $B_\epsilon^A(a) = A \cap B_\epsilon^Y(a) \subseteq A \cap U = O$.
Since O contains a ball in (A, d) around each of its points a, O is open in (A, d). •

More informally, we can say: an open set O in (A, d) is the "restriction to A" of an open set in (Y, d).

The same is true for closed sets:

Corollary Suppose $F \subseteq (Y, d)$ and that $F \subseteq A$. Then F is a closed set in (A, d) iff there is a closed set C is (Y, d) for which $F = A \cap C$.

Proof C is closed in (Y, d) iff
$Y - C$ is open in (Y, d) iff
$A \cap (Y - C)$ is open in (A, d) iff
$A - (A \cap (Y - C))$ is closed in (A, d) iff
$A \cap C$ is closed in (A, d) (*because* $A - (A \cap (Y - C)) = A \cap C$) •

When we have $A \subseteq (Y, d)$ and we give A the same pseudometric d as in Y, then we say that A is a <u>*subspace*</u> *(not just a sub*<u>*set*</u>*) of* (Y, d).

Corollary Suppose $O \subseteq A \subseteq (Y, d)$. <u>If A is open</u> in (Y, d) and O is open in (A, d), then O is open in (X, d).

Proof O is open in (A, d) iff $O = A \cap U$ where U is open in (Y, d). But if A is open in (Y, d), then $O = A \cap U$ is open in (Y, d).

More informally, an open set in an open subspace is open in the larger space.

You can prove a parallel statement about closed sets in closed subspaces.

In HW 4, #7:

i) It is assumed that the same metric d from (X, d) is being used to measure distances in all the O_α's and F_α's.

ii) It is handy to know how open sets in O_α, $\bigcup_{\alpha \in A} O_\alpha$, and X are related. That's where these comments may be handy.
For example to prove that $\bigcup_{\alpha \in A} f_\alpha = f : \bigcup_{\alpha \in A} O_\alpha \to Y$ isa continuous you need to know how an open sets in $\bigcup_{\alpha \in A} O$ are related to open sets in each separate $O_\alpha \subseteq \bigcup_{\alpha \in A} O_\alpha$.

iii) Parts b) and c) involve closed sets. So it should be easier to check continuity using the criterion that "f is continuous iff the inverse images of closed sets are closed"

$\bigcup_{\alpha \in A} f_\alpha = f : \bigcup_{\alpha \in A} O_\alpha \to Y$