Math 417, Fall 2009
Homework 5

Homework 4 will be <u>due in class on Tuesday, October 27</u>.

Problems to think about: <u>not</u> to hand in:

Prove or give a counterexample:

i) For any x in a space $(X, \mathcal{T})$, $\{x\}$ is equal to the intersection of all open sets containing x.

ii) In $(X, \mathcal{T})$, a finite set must be closed.

iii) If for each $\alpha \in A$, each $\mathcal{T}_\alpha$ is a topology on X, then $\bigcap\{\mathcal{T}_\alpha \colon \alpha \in A\}$ is a topology on X.

iv) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, then there is a unique smallest topology $\mathcal{T}_3$ on X such that $\mathcal{T}_3 \supseteq \mathcal{T}_1 \cup \mathcal{T}_2$.

v) If for each $\alpha \in A$, each $\mathcal{T}_\alpha$ is a topology on X, then there is a unique smallest topology $\mathcal{T}$ on X such that for each α, $\mathcal{T} \supseteq \mathcal{T}_\alpha$.

<u>To hand in</u>:

1. Let $X = \{0, 1, 2, 3, \ldots\}$. For $O \subseteq X$, let $|\, O \cap [1, n]\,|$ = the cardinality of $O \cap [1, n]$. Define

$$\mathcal{T} = \{\, O : \ 0 \notin O \ \text{ or } \ (0 \in O \ \text{ and } \lim_{n \to \infty} \tfrac{|\, O \cap [1,n]\,|}{n} = 1)\,\}.$$

a) Prove that $\mathcal{T}$ is a topology on X.

b) *In <u>any</u> space $(Y, \mathcal{T})$: x is called a <u>limit point</u> of a subset A if $N \cap (A - \{x\}) \neq \emptyset$ for every neighborhood N of x. Informally, this means that "there are points in A, different from x, that are arbitrarily close to x."*

Prove that in any space $(Y, \mathcal{T})$, a subset B is closed iff B contains all of its limit points.

c) For the space $(X, \mathcal{T})$ defined above, prove that x is a limit point of X if and only if $x = 0$.

2. A space $(X, \mathcal{T})$ is called a T_1-*space* if, for each $x \in X$, $\{x\}$ is closed.

a) Give an example of a space $(X, \mathcal{T})$ that is not a T_1-space and where $\mathcal{T}$ is <u>not</u> the trivial topology.

b) Prove that X is a T_1-space if and only if:

whenever $x \neq y$, there exist open sets U and V for which
$x \in U, y \notin U$ and $y \in V, x \notin V$
(*that is: each point is in an open set that does not contain the other point*)

c) Prove that a subspace of a T_1-space is a T_1-space.

3. A space $(X, \mathcal{T})$ is called a T_2-*space* (or <u>*Hausdorff space*</u>) if whenever $x, y \in X$ and $x \neq y$, then there exist disjoint open sets U and V with $x \in U$ and $y \in V$.

a) Give an example of a space $(X, \mathcal{T})$ which is a T_1-space but not a T_2-space.

b) Prove that a subspace of a Hausdorff space is Hausdorff.

4. We say that a space $(X, \mathcal{T})$ satisfies the <u>countable chain condition</u> ($=$ CCC) if every family of disjoint open sets must be countable.

a) Prove that a separable space $(X, \mathcal{T})$ satisfies the CCC.

b) Give an example of a space satisfying CCC which is not separable.

5. A point $x \in (X, \mathcal{T})$ is called a <u>condensation point</u> if every neighborhood of x is uncountable.

a) Prove that the set C of all condensation points in X is closed.

b) Prove that if X has a countable base $\mathcal{B}$ for the topology, then $X - C$ is countable.

6. Suppose that $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are topological spaces. The product topology on $X \times Y$ is defined to be the topology for which the collection of "open boxes"

$$\mathcal{B} = \{U \times V : U \in \mathcal{T},\ V \in \mathcal{T}'\} \text{ is a base.}$$

Therefore a set $O \subseteq X \times Y$ is open in the product topology iff for all $(x, y) \in O$, there are open sets $U \subseteq X$ and $V \subseteq Y$ such that $(x, y) \in U \times V \subseteq O$.

You should check for yourself that $\mathcal{B}$ does satisfy the necessary conditions in The Base Theorem. But don't include that with your solutions. We always assume that the topology on $X \times Y$ is the product topology unless something different is explicitly stated.

Note that the product topology on $\mathbb{R} \times \mathbb{R}$ is the usual topology on $\mathbb{R}^2$.

a) Prove that if X and Y are Hausdorff, then $X \times Y$ is Hausdorff. (*See #3.*)

b) Let $\pi_X : X \times Y \to X$ be the projection map $\pi_X(x, y) = x$. Prove that

i) if U is open in X, then $\pi_X^{-1}[U]$ is open in $X \times Y$

ii) if O is open in $X \times Y$, then $\pi_X[O]$ is open in X.
(*Caution: you cannot assume that O is a "box" $U \times V$!*)

i) and ii) say that π_X is a continuous, open map. The same results are also true for the other projection $\pi_Y : X \times Y \to Y$ defined by $\pi_Y(x, y) = y$.

c) Prove that if $A \subseteq X$ and $B \subseteq Y$, then $\mathrm{cl}_{X \times Y}(A \times B) = \mathrm{cl}_X A \times \mathrm{cl}_Y B$. Use this to prove that $X \times Y$ is separable if X and Y are separable.

Note: c) also implies that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$: "a product of closed sets is closed."

d) Suppose (X, d_1) and (Y, d_2) are pseudometric spaces. Define a pseudometric d on the set $X \times Y$ by

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$$

Prove that the product topology on $X \times Y$ is the same as the topology $\mathcal{T}_d$.

Note: This proves that the product of two pseudometric spaces is pseudometrizable.
d is the analogue of the taxicab metric in $\mathbb{R}^2$. Other pseudometrics on $X \times Y$ that are equivalent to d are

$$d'((x_1, y_1), (x_2, y_2)) = (d_1^2(x_1, x_2) + d_2^2(y_1, y_2))^{1/2}$$

$$d''((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$$