Math 417, Fall 2009
Homework 6

Homework 6 will be due in class on Thursday, November 5

Problems to think about: not to hand in:

i) For every possible topology $\mathcal{T}$, the space $(\{0,1,2\}, \mathcal{T})$ is pseudometrizable.
ii) Suppose $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and that for every subset A of X, $\mathrm{cl}_{\mathcal{T}}(A) = \mathrm{cl}_{\mathcal{T}'}(A)$. Then $\mathcal{T} = \mathcal{T}'$.
iii) If $f: X \to Y$ is both continuous and open, then f is also closed.
iv) If A and B are subspaces of $(X, \mathcal{T})$ and both A and B are discrete in the subspace topology, then $A \cup B$ is discrete in the subspace topology.
iv) Suppose $\mathcal{T}$ is the cofinite topology on X. Every bijection $f: (X, \mathcal{T}) \to (X, \mathcal{T})$ is a homeomorphism.
v) The Sorgenfrey plane has a subspace that is homeomorphic to $\mathbb{R}$ (with its usual topology).

To hand in:

1. a) Every countable metric space (X, d) space has a basis that consists of clopen sets.

b) Let (X, d) be a separable pseudometric space, and suppose that there is a basis $\mathcal{B}$ for $\mathcal{T}_d$ where all sets in $\mathcal{B}$ are clopen. Prove that (X, d) has a countable basis $\mathcal{B}'$ where the sets in $\mathcal{B}'$ are clopen.

Note: $\mathcal{B}$ might not be countable.
Hint: In (X, d) second countability, separability and the Lindelöf property are equivalent; so you have lots of properties you could use.

c) (Optional) *Can you generalize: do we need a pseudometric space (X, d) or would the same result be true for a wider class of topological spaces X ?*)

2. A space $(X, \mathcal{T})$ is called *hereditarily Lindelöf* if every subspace of X is Lindelöf.

a) Prove that a second countable space is hereditarily Lindelöf.

b) Suppose X is hereditarily Lindelöf.
Prove that set $A = \{x \in X : x \text{ is not a limit point of } X\}$ is countable.

3. Suppose X is an infinite set with the cofinite topology, and that Y is a T_1-space. Let $f: X \to Y$ be continuous and onto. Prove that either f is constant or X is homeomorphic to Y.

Note: The problem does not say that if f is not constant, then f is a homeomorphism.
Hint: First prove that if f is not constant, then $|X| = |Y|$. Then examine the topology of Y.)

4. Suppose $\mathcal{P}$ and $\mathcal{B}$ are two bases for the topology in $(X, \mathcal{T})$, and that $\mathcal{P}$ and $\mathcal{B}$ are infinite.

a) Prove that there is a subcollection $\mathcal{B}' \subseteq \mathcal{B}$ such that $\mathcal{B}'$ is also a base and $|\mathcal{B}'| \leq |\mathcal{P}|$. (*Hint: For each pair* $t = (U, V) \in \mathcal{P} \times \mathcal{P}$, *pick, if possible, a set* $W_t \in \mathcal{B}$ *such that* $U \subseteq W \subseteq V$*; otherwise set* $W_t = \emptyset$*.*)

b) Use part a) to prove that the Sorgenfrey line is not second countable. (*Hint: Show that otherwise there would have to be a countable base of sets of the form* $[a, b)$, *and then show that this is impossible.*)

5. A function $f : (X, \mathcal{T}) \to \mathbb{R}$ is called

$$\begin{cases} \underline{\text{lower semicontinuous}} \text{ if } f^{-1}[\,(b,\infty)\,] = \{x : f(x) > b\} \text{ is open for every } b \in \mathbb{R} \\ \underline{\text{upper semicontinuous}} \text{ if } f^{-1}[\,(-\infty, b)\,] = \{x : f(x) < b\} \text{ is open for every } b \in \mathbb{R} \end{cases}$$

a) Show that f is continuous iff f is both upper and lower semicontinuous.

b) Give an example of a lower semicontinuous $f : \mathbb{R} \to \mathbb{R}$ which is not continuous. Do the same for upper semicontinuous.

c) Prove that the characteristic function χ_A of a set $A \subseteq (X, \mathcal{T})$ is lower semicontinuous if A is open in X and upper semicontinuous if A is closed in X.

6. Consider a countable set X with the cofinite topology $\mathcal{T}$. State and prove a theorem that completely answers the question: "In $(X, \mathcal{T})$, what sequences converge to what points?

7. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces and $f\colon X \to Y$. Let

$$\Gamma(f) = \{(x, y) \in X \times Y : y = f(x)\} \subseteq X \times Y$$

Define $h\colon X \to \Gamma(f)$ by $h(x) = (x, f(x))$

Prove that h is a homeomorphism if and only if f is continuous.

Note: $\Gamma(f)$ *is usually called the "graph of* f*" , so the problem states that the graph of function is homeomorphic to its domain iff the function is continuous.*
(*Of course, if we think of* f *as a set of ordered pairs, then the graph of* f <u>*is*</u> f.)