Suppose \( A \) is an uncountable subset of \( \mathbb{R} \) with \( |A| = m < c \). (Of course, there is no such set \( A \) if the Continuum Hypothesis is assumed.) Is it possible for \( A \) to be closed? Explain.

1. Prove that in a metric space \((X, d)\), the following are equivalent:
   a) every Cauchy sequence is eventually constant
   b) \((X, d)\) is complete and \( T_d \) is the discrete topology
   c) for every \( A \subseteq X \), each Cauchy sequence in \( A \) converges to a point in \( A \)
      (that is, every subspace of \((X, d)\) is complete).

2. a) Suppose \( X \) is a Hausdorff space and that \( f : X \to X \) is continuous. Prove that the set of all fixed points \( C = \{ x \in X : f(x) = x \} \) is closed.

   b) Suppose \( A \subseteq \mathbb{R} \) and that \( A \) is closed. Prove that there exists a continuous \( f : \mathbb{R} \to \mathbb{R} \) such that \( A = \{ x \in \mathbb{R} : f(x) = x \} \).

   \[ \text{Hint: One way is this. Begin by finding a function } g : \mathbb{R} \to \mathbb{R} \text{ such that } g(x) = 1 \text{ iff } x \in A. \]
   \[ \text{You should be able to define such a } g \text{ in terms of the function } d(x, A). \text{ Or perhaps you have a better idea. Once you have such a } g, \text{ then you might get an idea from part of Example 4.2.1.)} \]

3. a) Suppose \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and that there is a constant \( K < 1 \) such that \( |f'(x)| \leq K \) for all \( x \). Prove that \( f \) is a contraction (and therefore has a unique fixed point.)

   b) Give an example of a continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that
   \[ |f(x) - f(y)| < |x - y| \]
   for all \( x \neq y \in \mathbb{R} \) but such that \( f \) has no fixed point.

   \[ \text{Note: The function } f \text{ is not a contraction mapping. The contraction mapping theorem would not be true if we allowed } \alpha = 1 \text{ in the definition of contraction mapping.} \]
4. Let \( f : (X, d) \to (X, d) \), where \((X, d)\) is a nonempty complete metric space. Let \( f^k \) denote the “\( k \)th iterate of \( f \)” — that is, \( f \) composed with itself \( k \) times.

a) Suppose that \( \exists k \in \mathbb{N} \) for which \( f^k \) is a contraction. Then, by the Contraction Mapping Theorem, \( f^k \) has a unique fixed point \( p \). Prove that \( p \) is also the unique fixed point for \( f \).

b) Prove that the function \( \cos : \mathbb{R} \to \mathbb{R} \) is not a contraction.

c) Prove that \( \cos^k \) is a contraction for some \( k \in \mathbb{N} \).

(Hint: the Mean Value Theorem may be helpful.)

d) Let \( k \in \mathbb{N} \) be such that \( g = \cos^k \) is a contraction and let \( p \) be the unique fixed point of \( g \). By a), \( p \) is also the unique solution of the equation \( \cos x = x \). Start with 0 as a “first approximation” for \( p \) and use the technique in the proof of the Contraction Mapping Theorem to find an \( n \in \mathbb{N} \) so that \( |g^n(0) - p| < 0.00001 \).

e) For this \( n \), use a calculator or computer to evaluate \( g^n(0) \). (This “solves” the equation \( \cos x = x \) with \( |\text{Error}| < 0.00001 \).)

5. Consider the differential equation \( y' = x - y \) with the initial condition \( y(0) = 2 \). Choose a suitable rectangle \( D \) and suitable constants \( K, M \) and \( a \) as in the proof of Picard's Theorem. Use the technique in the proof of the contraction mapping theorem to find a solution for the initial value problem. Identify the interval \( I \) in the proof. Is the solution you found actually valid on an interval larger than \( I \)?