Math 417, Fall 2009
Homework 7

Homework 7 will be <u>due in class on Thursday, November 19</u>

<u>Not to hand in</u>

Suppose A is an uncountable subset of $\mathbb{R}$ with $|A| = m < c$. (*Of course, there is no such set A if the Continuum Hypothesis is assumed.*) Is it possible for A to be closed? Explain.

1. Prove that in a metric space (X, d), <u>the following are equivalent</u>:

a) every Cauchy sequence is eventually constant
b) (X, d) is complete and $\mathcal{T}_d$ is the discrete topology
c) for every $A \subseteq X$, each Cauchy sequence in A converges to a point in A
(that is, every subspace of (X, d) is complete).

2. a) Suppose X is a Hausdorff space and that $f : X \to X$ is continuous. Prove that the set of all fixed points $C = \{x \in X : f(x) = x\}$ is closed.

b) Suppose $A \subseteq \mathbb{R}$ and that A is closed. Prove that there exists a continuous $f : \mathbb{R} \to \mathbb{R}$ such that $A = \{x \in \mathbb{R} : f(x) = x\}$.

Hint: One way is this. Begin by finding a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 1$ iff $x \in A$. You should be able to define such a g in terms of the function $d(x, A)$. Or perhaps you have a better idea. Once you have such a g, then you might get an idea from part of Example 4.2.1.)

3. $a)$ Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there is a constant $K < 1$ such that $|f\,'(x)| \leq K$ for all x. Prove that f is a contraction (and therefore has a unique fixed point.)

b) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that
$$|\,f(x) - f(y)\,| < |x - y\,|$$
for all $x \neq y \in \mathbb{R}$ but such that f has no fixed point.

Note: The function f is not a contraction mapping. The contraction mapping theorem would not be true if we allowed $\alpha = 1$ in the definition of contraction mapping.

4. Let $f\colon (X,d) \to (X,d)$, where (X,d) is a nonempty complete metric space. Let f^k denote the "k^{th} iterate of f" – that is, f composed with itself k times.

a) Suppose that $\exists k \in \mathbb{N}$ for which f^k is a contraction. Then, by the Contraction Mapping Theorem, f^k has a unique fixed point p. Prove that p is also the unique fixed point for f.

b) Prove that the function $\cos : \mathbb{R} \to \mathbb{R}$ is not a contraction.

c) Prove that $\cos^k$ is a contraction for some $k \in \mathbb{N}$.
(*Hint: the Mean Value Theorem may be helpful.*)

d) Let $k \in \mathbb{N}$ be such that $g = \cos^k$ is a contraction and let p be the unique fixed point of g. By a), p is also the unique solution of the equation $\cos x = x$. Start with 0 as a "first approximation" for p and use the technique in the proof of the Contraction Mapping Theorem to find an $n \in \mathbb{N}$ so that $|\,g^n(0) - p\,| < 0.00001$.

e) For this n, use a calculator or computer to evaluate $g^n(0)$. (*This "solves" the equation* $\cos x = x$ *with* $|\,Error\,| < 0.00001$.)

5. Consider the differential equation $y' = x - y$ with the initial condition $y(0) = 2$. Choose a suitable rectangle D and suitable constants K, M and a as in the proof of Picard's Theorem. Use the technique in the proof of the contraction mapping theorem to find a solution for the initial value problem. Identify the interval I in the proof. Is the solution you found actually valid on an interval larger than I ?