Homework 8 will be due in class on Thursday, December 3.

1. Let $T$ be the cofinite topology on $X$. Prove that $(X, T)$ is a Baire space if and only if $X$ is either finite or uncountable.

2. Suppose that $(X, d)$ is a complete metric space and that $F$ is a family of continuous functions from $X$ to $\mathbb{R}$ with the following property:

\[ \forall x \in X, \exists \text{ a constant } M_x \text{ such that } |f(x)| \leq M_x \text{ for all } f \in F \]

Prove that there exists a nonempty open set $U$ and a constant $M$ (independent of $x$) such that $|f(x)| \leq M$ for all $x \in U$ and all $f \in F$.

*Hint: Let $E_k = \{ x \in X : |f(x)| \leq k \text{ for all } f \in F \}$. Use the Baire Category Theorem.*

3. Suppose that $(X, T)$ is a topological space and $f : X \to Y$, where $(Y, d)$ is a metric space. For $x \in X$, define

\[ \omega_f(x) = \text{"the oscillation of } f \text{ at } x = \inf \{ \text{diam}(f[N]) : N \text{ is a neighborhood of } x \} \]

a) Prove that $f$ is continuous at $a$ if and only if $\omega_f(a) = 0$.

b) Prove that for $n \in \mathbb{N}$, $\{ x \in X : \omega_f(x) < \frac{1}{n} \}$ is open in $X$.

c) Prove that $\{ x \in X : f \text{ is continuous at } x \}$ is a $G_\delta$-set in $X$.

*As mentioned in class: $\mathbb{Q}$ is not a $G_\delta$ set in $\mathbb{R}$. By c), there cannot exist a function $f : \mathbb{R} \to \mathbb{R}$ for which $\mathbb{Q} = \{ x \in \mathbb{R} : f \text{ is continuous at } x \}$; equivalently, there cannot exist a function $f : \mathbb{R} \to \mathbb{R}$ for which the set of discontinuities is $\mathbb{Q}$.*

4. Suppose $(X, T)$ is compact and that $f : (X, T) \to (Y, T')$ is continuous and onto. Prove that $(Y, T')$ is compact.

5. a) Suppose that $(X, T)$ is compact and $(Y, S)$ is Hausdorff. Let $f : (X, T) \to (Y, S)$ be a continuous bijection. Prove that $f$ is a homeomorphism.

b) Let $T$ be the usual topology on $[0, 1]$ and suppose $T_1$ and $T_2$ are two other topologies on $[0, 1]$ such that $T_1 \subset T \subset T_2$ and $([0, 1], T_1)$ is not Hausdorff and that $([0, 1], T_2)$ is not compact.

*Hint: Consider the identity map $i : [0, 1] \to [0, 1].$*

c) If part b) true if $[0, 1]$ is replaced by an arbitrary compact Hausdorff space $(X, T)$?

(OVER) →
6. Suppose that $A$ and $B$ are disjoint nonempty closed sets in $(X, d)$ and that $A$ is compact. Prove that $d(A, B) > 0$.

7. Let $(X, T)$ and $(Y, S)$ be topological spaces.
   
   a) Prove that $X$ is compact iff every open cover by basic open sets has a finite subcover.
   
   b) Suppose $X \times Y$ is compact. Prove that if $X, Y \neq \emptyset$, then $X$ and $Y$ are compact.
   (By induction, a similar statement applies to any finite product.)

   c) Prove that if $X$ and $Y$ are compact, then $X \times Y$ is compact. (By induction, a similar statement applies to any finite product.)
   (Hint: for any $x \in X$, $\{x\} \times Y$ is homeomorphic to $Y$. Part a) is also relevant.)