**Math 417, Fall 2009**
**Homework 8**

Homework 8 will be <u>due in class on Thursday, December 3</u>

1. Let $\mathcal{T}$ be the cofinite topology on $X$. Prove that $(X, \mathcal{T})$ is a Baire space if and only if $X$ is either finite or uncountable.

2. Suppose that $(X, d)$ is a complete metric space and that $\mathcal{F}$ is a family of continuous functions from $X$ to $\mathbb{R}$ with the following property:

$$\forall x \in X,\ \exists \text{ a constant } M_x \text{ such that } |f(x)| \leq M_x \text{ for all } f \in \mathcal{F}$$

Prove that there exists a nonempty open set $U$ and a constant $M$ (independent of $x$) such that $|f(x)| \leq M$ for all $x \in U$ and all $f \in \mathcal{F}$.

*Hint: Let $E_k = \{x \in X \colon |f(x)| \leq k$ for all $f \in \mathcal{F}\}$. Use the Baire Category Theorem.*

3. Suppose that $(X, \mathcal{T})$ is a topological space and $f \colon X \to Y$, where $(Y, d)$ is a metric space. For $x \in X$, define

$$\omega_f(x) \ = \text{ “the oscillation of } f \text{ at } x\text{”} = \inf\{\text{diam}(f[N]) \colon N \text{ is a neighborhood of } x\}$$

a) Prove that $f$ is continuous at $a$ if and only if $\omega_f(a) = 0$.
b) Prove that for $n \in \mathbb{N}$, $\{x \in X : \omega_f(x) < \frac{1}{n}\}$ is open in $X$.
c) Prove that $\{x \in X \colon f$ is continuous at $x\}$ is a $G_\delta$-set in $X$.

*As mentioned in class: $\mathbb{Q}$ is not a $G_\delta$ set in $\mathbb{R}$. By c), there cannot exist a function $f : \mathbb{R} \to \mathbb{R}$ for which $\mathbb{Q} = \{x \in \mathbb{R} : f$ is continuous at $x\}$; equivalently, there cannot exist a function $f : \mathbb{R} \to \mathbb{R}$ for which the set of discontinuities is $\mathbb{P}$.*

4. Suppose $(X, \mathcal{T})$ is compact and that $f : (X, \mathcal{T}) \to (Y, \mathcal{T}\,')$ is continuous and onto. Prove that $(Y, \mathcal{T}\,')$ is compact.

5. a) Suppose that $(X, \mathcal{T})$ is compact and $(Y, \mathcal{S})$ is Hausdorff. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ be a continuous bijection. Prove that $f$ is a homeomorphism.

b) Let $\mathcal{T}$ be the usual topology on $[0, 1]$ and suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two other topologies on $[0, 1]$ such that $\mathcal{T}_1 \underset{\neq}{\subseteq} \mathcal{T} \underset{\neq}{\subseteq} \mathcal{T}_2$ $([0, 1], \mathcal{T}_1)$ is not Hausdorff and that $([0, 1], \mathcal{T}_2)$ is not compact.

(*Hint: Consider the identity map* $i :\ [0, 1] \to [0, 1]$. )

c) If part b) true if $[0, 1]$ is replaced by an arbitrary compact Hausdorff space $(X, \mathcal{T})$?

**(OVER) →**

6. Suppose that $A$ and $B$ are disjoint nonempty closed sets in $(X, d)$ and that $A$ is compact. Prove that $d(A, B) > 0$.

7. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces.

a) Prove that $X$ is compact iff every open cover by basic open sets has a finite subcover.

b) Suppose $X \times Y$ is compact. Prove that if $X, Y \neq \emptyset$, then $X$ and $Y$ are compact. (*By induction, a similar statement applies to any finite product.*)

c) Prove that if $X$ and $Y$ are compact, then $X \times Y$ is compact. (*By induction, a similar statement applies to any finite product.*)
(*Hint: for any $x \in X$, $\{x\} \times Y$ is homeomorphic to $Y$. Part a) is also relevant.*)