# CLASSROOM NOTES

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## A REGULAR SPACE, NOT COMPLETELY REGULAR

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In 1930 Tychonoff [1] gave an example of a regular space that is not completely regular. Hewitt [2] and Novak [3] modified this example to obtain spaces in which every continuous real valued function is constant. Authors of topology textbooks generally cite one of these papers as evidence that, while regular spaces which are not completely regular exist, they are quite complicated. I know of no textbook which goes through the details of such an example. (The example on page 154 of Dugundji's *Topology* is not regular, as noted in [2].) The following example of such a space, which I believe is particularly transparent, might be of interest.

Consider the following subsets of the Cartesian plane. For a fixed even integer $n$, $L_n$ is the set of points $(n, y)$ with $0 \leqq y < 1/2$. $S_1$ is the union of the sets $L_n$. For a fixed odd integer $n$, and integer $k \geqq 2$, $p_{n,k} = (n, 1-1/k)$, and $T_{n,k}$ is the set of points of the form $(n \pm t, 1-t-1/k)$ as $t$ ranges over the interval $(0, 1-1/k]$. These are the points on the legs of an isosceles right triangle with hypotenuse lying along the $x$-axis, and $p_{n,k}$ the vertex at the right angle. $S_2$ is the set of all $p_{n,k}$ and $S_3$ is the union of the sets $T_{n,k}$.

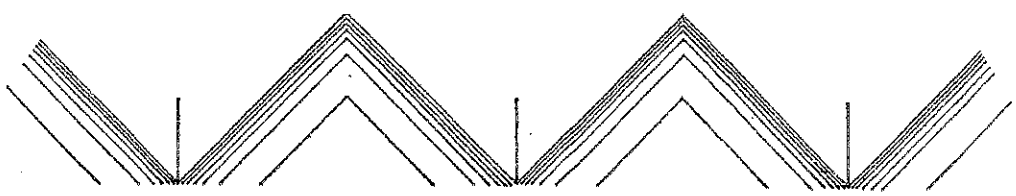

FIG. 1

The underlying set of our topological space $X$ (which is sketched in Fig. 1) is the union of $S_1$, $S_2$, and $S_3$, plus two additional points $p_-$ and $p_+$. We define its topology by specifying the neighborhoods of each point. The topology is discrete at each point of $S_3$. A neighborhood of the point $p_{n,k}$ must contain all but finitely many points of $T_{n,k}$. A neighborhood of a point $(n, y)$ of $L_n$ contains all but finitely many of the points in $X$ which have the same ordinate $y$, and an abscissa which differs from $n$ by less than 1. (In other words, draw a horizontal line segment extending one unit to the left and right. A neighborhood contains all but a finite number of the points of intersection of the segment with the legs of the triangles $T_{n,k}$.) A neighborhood of $p_-$ contains all the points with abscissas less than some real number $c$, and a neighborhood of $p_+$ contains all the points with abscissas greater than some real number $c$.

A collection $\mathcal{F}$ of sets is called a filter if (1) $\emptyset \notin \mathcal{F}$ (2) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$ (3) $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$.

Example (and "prototype" for the definition): In a space $X$, the neighborhood system $\mathcal{N}_x$ at $x$ is a filter.



If one is given for each point $x$ of a set $X$, a filter $F_x$ of subsets containing $x$, and defines $\tau$ to be the collection of sets $U$ such that $U$ belongs to $F_x$ for each $x$ in $U$, then it is well known that (1) $\tau$ is a toplogy, and (2) for each $x$ the filter of $\tau$-neighborhoods is contained in $F_x$, and is equal to $F_x$ provided that this compatibility property holds: If $U$ belongs to $F_x$, there is a $W$ contained in $F_x$, such that $U$ belongs to $F_y$ for each $y$ in $W$.

It is easily checked that the filters just specified satisfy this property, hence are indeed the neighborhood systems in the topology they induce. It is trivial to check the neighborhood system at each point has a basis of closed sets, thus $X$ is regular. $X$ fails to be completely regular since, as we shall see, every real continuous function must take the same value at $p_-$ and $p_+$.

This just says that the neighborhoods described give a topology with the nbds at each point as given. And that this topology is $T_3$.

In any topological space it is true that the set on which a continuous real function agrees with its value at the point $x$, is the intersection of the countable family of neighborhoods $N_j$ of $x$ ($j=1, 2, \cdots$) on which $f$ differs from its value at $x$ by less than $1/j$. It follows that for fixed $n$ and $k$, the set of "anomalous" points of $T_{n,k}$ at which $f$ fails to be equal to $f(p_{n,k})$ is countable. Let us denote the set of ordinates of these anomalous points by $S_{n,k}$, and let $S_n = \bigcup_k S_{n,k}$. Now pick a point $p$ of $L_{n-1}$ or $L_{n+1}$ whose ordinate does not belong to the countable set $S_n$. Clearly

$S_{nk} = \{y : (x,y) \in T_{n,k}$ and $f(x,y) \neq f(p_{n,k})\}$ ($n$ odd)

$$f(p) = \lim_{k\to\infty} f(p_{n,k}) = c_n.$$

As $f$ takes the value $c_{n+1}$ and $c_{n-1}$ at all but countably many points of $L_n$, it must be that $c_n = c$ for all $n$. Then $f(p_-) = f(p_+) = c$ since $f$ assumes the value $c$ in every neighborhood of both points.

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## References

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# A REGULAR SPACE ON WHICH EVERY CONTINUOUS REAL-VALUED FUNCTION IS CONSTANT

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The well-known examples, due to Hewitt [2] and Novák [3], of the type of spaces mentioned in the title of this article are not easy to present in an introductory level course of topology because their constructions depend heavily on the theory of cardinal numbers. More recently, Herrlich [1] outlined an easier construction of such a space, but his construction also involves many properties of cardinal numbers. The purpose of this note is to outline the construction of such a space that avoids all cardinality arguments except the distinction between countable and uncountable sets.

The space with which we begin the construction is a regular space $Q$ that has two points $p^-$ and $p^+$ such that $f(p^-)=f(p^+)$ for every continuous real-valued function $f$ on $Q$. Thomas [4] recently gave a simple geometric construction of such a space using only the distinction between countable and uncountable sets. The construction of such a space is the most difficult part of Herrlich's paper, and it is the part that depends on cardinal numbers. One can now proceed *verbatim* with the construction as outlined in Herrlich's paper. For completeness we include these steps:

PROPOSITION: *For any regular space $Z$, there exists a regular space $Q(Z)$ such that $Z$ is imbedded as a subspace of $Q(Z)$ and every continuous real-valued function of $Q(Z)$ is constant on $Z$.*

To obtain $Q(Z)$, one first constructs a topology on the product set $Z\times Q$ by declaring a subset $V\subset Z\times Q$ to be open if the following are true:

1. If $(z, x)\in V$, then there is a neighborhood $U$ of $x$ in $Q$ such that $\{z\}\times U\subset V$.

2. If $(z, p^+)\in V$, then there is a neighborhood $U$ of $z$ in $Z$ such that $U\times\{p^+\}\subset V$.

With this topology $Z\times Q$ is regular and $Z$ is homeomorphic to the line $Z\times\{p^+\}$ in $Z\times Q$. It should be pointed out that the neighborhoods on the upper edge $Z\times\{p^+\}$ of the square $Z\times Q$ are somewhat complicated; one may picture them as appearing like "ragged-edged combs." The space $Q(Z)$ is now obtained by identifying all the points of the line $Z\times\{p^-\}$ in $Z\times Q$.

In order to complete the construction, let $X_0$ be a one-point space, and inductively, let $X_{n+1}=Q(X_n)$ for all positive integers $n$. Thus, an increasing sequence $X_0\subset X_1\subset\cdots\subset X_n\subset\cdots$ of regular spaces is obtained. Let $X=\bigcup_{n=0}^{\infty}X_n$ and define a topology on $X$ by declaring a set $U\subset X$ to be open if $U\cap X_n$ is open in $X_n$ for each $n$. The space $X$ thus obtained has the desired properties.

## References

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