Math 418, Spring 2010
Homework 1

Homework 1 is due in class on Thursday, January 28

1. Let (X, d) and (Y, d') be two unbounded connected metric spaces. Suppose $k > 0$ and that $(a, b) \in X \times Y$. Let $K = \{(x, y) \in X \times Y : d(x, a) \leq k$ and $d'(y, b) \leq k\}$. Prove that the complement of K in $X \times Y$ is connected.
Hint: Try to prove it first for the case $X = Y = \mathbb{R}$ and $d = d' =$ the usual metric.

2. a) Find the cardinality of the collection of all compact connected subsets of $\mathbb{R}^2$.
b) Find the cardinality of the collection of all connected subsets of $\mathbb{R}^2$.

3. Suppose (X, d) is a connected metric space with $\mid X \mid > 1$. Prove that $\mid X \mid \geq c$.
Hint: think about some continuous functions from X to $\mathbb{R}$.

4. A metric space (X, d) satisfies the <u>ϵ-chain</u> <u>condition</u> if for all $\epsilon > 0$ and all x, $y \in X$, there exists a finite set of points x_1, x_2, ... , x_{n-1}, x_n where $x_1 = x$, $x_n = y$, and $d(x_i, x_{i+1}) < \epsilon$ for all $i = 1$, ..., $n - 1$.

a) Give an example of a metric space which satisfies the ϵ-chain condition but which is not connected.
b) Prove that if (X, d) is connected, then (X, d) satisfies the ϵ-chain condition.
c) Prove that if (X, d) is compact and satisfies the ϵ-chain condition, then X is connected.
d) Prove that $(\{0\} \times [-1, 1]) \cup \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\} \subseteq \mathbb{R}^2$ is connected.
Use c) to give a different proof than the one given in Example 3.4

5. Prove that there does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f[\mathbb{P}] \subseteq \mathbb{Q}$ and $f[\mathbb{Q}] \subseteq \mathbb{P}$.
Hint: One method: What do you know about ran f? What else do you know?
Another method: if such an f exists, let $g = \frac{1}{1+|f|}$ and let $h = g|[0, 1]$. What do you know about h?

6. Let $A = \mathbb{Q} \cap [0, 1]$ and $p = (0, 1) \in \mathbb{R}^2$. Let $X = \{(x, y) \in \mathbb{R}^2 :$ (x, y) lies on a line segment joining p to a point in $A\}$.

a) Prove that X is path connected
b) Prove that the condition in Definition 3.5a) is satisfied in X only at $x = p$ (so X is not locally connected).

(over $\longrightarrow$)

7. a) Prove that for any space X and $n \geq 2$,

> if X has $\geq n$ components, then there are nonempty pairwise separated sets $H_1, ..., H_n$ for which $X = H_1 \cup ... \cup H_n$ (**)

Hint. For a given n, do not start with the components and try to group them to form the H_n's. Start with the fact that X is not connected. Use induction.

b) Recall that a <u>disconnection</u> of X means a pair of nonempty separated sets A, B for which $X = A \cup B$. Remember also that if C is a component of X, C is not necessarily "one piece in a disconnection of X" *(see Example 4.4)*.

Prove that X has only finitely many components $n\,(n \geq 2)$ iff X has only finitely many disconnections.

Hint: When X has infinitely many components, then X has $\geq$ n components for every n.