Math 418, Spring 2010
Homework 2

Homework 2 is due in class on Tuesday, February 9

1. a) Suppose X and Y are topological spaces and that $A \subseteq X$, $B \subseteq Y$. Prove that $\text{int}_{X\times Y}(A \times B) = \text{int}_X A \times \text{int}_Y B$: that is, "the interior of the product is the product of the interiors." (*By induction, the same result holds for any* <u>*finite*</u> *product*.) Give an example to show that the statement may be false for <u>infinite</u> products.

b) Suppose $A_\alpha \subseteq X_\alpha$ for all $\alpha \in A$. Prove that in the product $X = \prod X_\alpha$,

$$\text{cl}\left(\prod A_\alpha\right) = \prod \text{cl}\, A_\alpha.$$

Note: When the A_α's are closed, this shows that $\prod A_\alpha$ is closed: so "any product of closed sets is closed." Can you see any plausible reason why products of closures are better behaved than products of interiors?

c) Suppose $\prod X_\alpha \neq \emptyset$ and that $A_\alpha \subseteq X_\alpha$. Prove that $\prod A_\alpha$ is dense in X iff A_α is dense in X_α for each α. *Note: Part c) implies that a finite product of separable spaces is separable. Part c) doesn't tell us whether or not an infinite product of separable spaces is separable: why not?*

d) For each α, let $q_\alpha \in X_\alpha$. Prove that $B = \{x \in \prod X_\alpha : x_\alpha = q_\alpha$ for all but at most finitely many $\alpha\}$ is dense in $\prod X_\alpha$. (*Note: Suppose $X = \mathbb{R}^{\mathbb{N}} = \prod_{n\in\mathbb{N}} X_n$ where each $X_n = \mathbb{R}$. Suppose each q_n is chosen to be a rational – say $q_n = 0$. What does d) say about $\mathbb{R}^{\mathbb{N}}$?*)

2. Let X be a topological space and consider the "diagonal" set

$$\Delta = \{(x,x) : x \in X\} \subseteq X \times X.$$

a) Prove that Δ is closed in $X \times X$ iff X is Hausdorff.
b) Prove that Δ is open in $X \times X$ iff X is discrete.

3. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent sequence of real numbers. A series $\sum_{n=1}^{\infty} a'_n$ where each $a'_n = a_n$ or $a'_n = 0$ is called a <u>subseries</u> of $\sum_{n=1}^{\infty} a_n$.

Prove that $S = \{s \in \mathbb{R} :\ s$ is the sum of a subseries of $\sum_{n=1}^{\infty} a_n\}$ is closed in $\mathbb{R}$.

Hint: "Absolute convergence" just guarantees that every subseries converges. Each subseries $\sum_{n=1}^{\infty} a'_n$ can be associated in a natural way with a point $x \in \{0,1\}^{\aleph_0}$. Consider the mapping $f : \{0,1\}^{\aleph_0} \to \mathbb{R}$ given by $f(x) = \sum_{n=1}^{\infty} a'_n \in \mathbb{R}$. Must f be a homeomorphism?

4. Let $X = [0,1]^{[0,1]}$ with the product topology.

a) Prove that the set of all functions in X with finite range (*sometimes called step functions*) is dense in X.

b) By Theorem 3.5, X is separable. Describe a countable set of step functions which is dense in X.

c) Let A be the set of points in X which are the characteristic functions of singleton sets $\{r\} \subseteq [0,1]$. Prove that A, with the subspace topology, is discrete and not separable.

5. "Boxes" of the form $\prod\{U_\alpha : \alpha \in A\}$, where U_α is open in X_α, are a base for the box topology on $\prod\{X_\alpha : \alpha \in A\}$. <u>Throughout this problem, we assume that products have the box topology rather than the usual product topology</u>.

a) Show that the "diagonal map" $f: \mathbb{R} \to \mathbb{R}^{\aleph_0}$ given by $f(x) = (x, x, x, ...)$ is not continuous, but that its composition with each projection map is continuous.

b) Show that $[0,1]^{\aleph_0}$ is not compact.

Hint: let $A_0 = [0,1)$ and $A_1 = (0,1]$. Consider the collection $\mathcal{U}$ of all sets of the form $A_{\epsilon_1} \times A_{\epsilon_2} \times ... \times A_{\epsilon_n} \times ...$, where $(\epsilon_1, \epsilon_2, ..., \epsilon_n, ...) \in \{0,1\}^{\aleph_0}$.

Note: In contrast, with the product topology, $[0,1]^m$ <u>is</u> compact for any cardinal m – by the Tychonoff Product Theorem (3.10) which we will prove later.

c) Show that $\mathbb{R}^{\aleph_0}$ is not connected by showing that the set $A = \{x \in \mathbb{R}^{\aleph_0} : x$ is an unbounded sequence in $\mathbb{R}\}$ is clopen.

d) Suppose (X, d) and (X_α, d_α) $(\alpha \in A)$ are metric spaces. Prove that a function $f : X \to \prod X_\alpha$ (with the <u>box</u> topology) is continuous iff each coordinate function $f_\alpha = \pi_\alpha \circ f$ is continuous <u>and</u> each $x \in X$ has a neighborhood on which all but a finite number of the f_α's are constant.