Math 418, Spring 2010
Homework 3

Homework 3 is due in class on Thursday, February 18.

1. A space $(X, \mathcal{T})$ is called a T_0 space if whenever $x \neq y \in X$, then $\mathcal{N}_x \neq \mathcal{N}_y$ (equivalently, either there is an open set U containing x but not y, or vice-versa). Notice that the T_0 condition is weaker than T_1 (*see example III.2.6.4).* Clearly, a subspace of a T_0-space is T_0.

a) Prove that a nonempty product $X = \prod\{X_\alpha\colon \alpha \in A\}$ is T_0 iff each X_α is T_0.

b) Let S be "Sierpinski space" – that is, $S = \{0, 1\}$ with the topology $\mathcal{T} = \{\emptyset, \{1\}, \{0, 1\}\}$. Use the embedding theorems to prove that a space X is T_0 iff X is homeomorphic to a subspace of S^m for some cardinal m. (*Nearly all interesting spaces are T_0, so nearly all interesting spaces are (topologically) just subspaces of S^m for some cardinal m.*)

Hint $\Rightarrow$: for each open set U in X, let χ_U be the characteristic function of U. Use an embedding theorems.

2. a) Let (X, d) be a metric space. Prove that $C(X)$ (= the collection of continuous real-valued functions on X) separates points from closed sets. (*Since X is T_1, $C(X)$ therefore also separates points*).

b) Suppose X is any T_0 topological space for which $C(X)$ separates points and closed sets. Prove that X can be embedded in a product of copies of $\mathbb{R}$.

3. A space X is called totally disconnected every connected subset A satisfies $|A| \leq 1$ (equivalently, if all components in X are singletons). Prove that a totally disconnected compact Hausdorff space is homeomorphic to a closed subspace of $\{0, 1\}^m$ for some m.

Hint: see Lemma V.5.6.

4. a) Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ given by $(x_1, y_1) \sim (x_2, y_2)$ iff $y_1 = y_2$. Prove that $\mathbb{R}^2/\sim$ is homeomorphic to $\mathbb{R}$.

b) Find a counterexample to the following assertion: if $\sim$ is an equivalence relation on a space X and each equivalence class is homeomorphic to the same space Y, then $(X/\sim) \times Y$ is homeomorphic to X.

Why might someone ever wonder whether this assertion might be true? In part a), we have $X = \mathbb{R} \times \mathbb{R}$, each equivalence class is homeomorphic to $\mathbb{R}$ and $(X/\sim) \times \mathbb{R} \simeq \mathbb{R} \times \mathbb{R} \simeq X$. In this example, you "divide out" equivalence classes that all look like $\mathbb{R}$, then "multiply" by $\mathbb{R}$, and you're back where you started.

c) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x, y) = x^2 + y^2$. Then the quotient space $\mathbb{R}^2/g$ is homeomorphic to what familiar space?

5. For $x, y \in [0,1]$, define $x \sim y$ iff $x - y$ is rational. Prove that the corresponding quotient space $[0,1]/\sim$ is trivial.

6. Suppose X_s (s $\in S$) and Y_t ($t \in T$) are pairwise disjoint spaces. Prove that $\sum_{s\in S} X_s \times \sum_{t\in T} Y_t$ is homeomorphic to $\sum_{s\in S, t\in T} (X_s \times Y_t)$.

7. A <u>base for the closed sets</u> in $(X, \mathcal{T})$ is a collection of $\mathcal{F}$ of <u>closed</u> sets such that every closed set F is an intersection of sets from $\mathcal{F}$. Clearly, $\mathcal{F}$ is a base for the closed sets in X iff $\mathcal{B} = \{O : O = X - F,\ F \in \mathcal{F}\}$ is a base for the open sets in X.

For a polynomial P in n real variable, define the <u>zero set</u> of P as

$$Z(P) = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n :\ P\ (x_1, x_2, ..., x_n)\ = 0\}$$

a) Prove that $\{Z(P) :\ P$ a polynomial in n real variables$\}$ is the base for the closed sets of a topology (called the <u>Zariski</u> topology) on $\mathbb{R}^n$.

b) Prove that the Zariski topology on $\mathbb{R}^n$ is T_1 but not T_2.

c) Prove that the Zariski topology on $\mathbb{R}$ is the cofinite topology, but that if $n > 1$, the Zariski topology on $\mathbb{R}^n$ is <u>not</u> the cofinite topology.

Note: The Zariski topology arises in studying algebraic geometry. After all, the sets $Z(P)$ are rather special geometric objects—those "surfaces" in $\mathbb{R}^n$ which can be described by polynomial equations $P\ (x_1, x_2, ..., x_n)\ = 0$.