Math 418, Spring 2010
Homework 4

Homework 4 is due in class on Tuesday, March 2.

1. Suppose X is a connected Tychonoff space with more than one point. Prove $|X| \geq c.$.

2. Prove that in any space X, a countable union of cozero sets is a cozero set $-$ equivalently, that a countable intersection of zero sets is a zero set.

3. Let $i : \mathbb{R} \to \mathbb{R}$ be the identity map and let

$$(i) = \{f \in C(\mathbb{R}) : f = gi \text{ for some } g \in C(\mathbb{R})\}.$$

For those who know a bit of algebra: $C(\mathbb{R})$ *(or, more generally,* $C(X)$ *) with addition and multiplication defined pointwise, is a commutative ring with unit.* (i) *is called the* *ideal* *generated by the element* i *.*

a) Prove that $(i) = \{f \in C(\mathbb{R}) : f(0) = 0 \text{ and the derivative } f'(0) \text{ exists}\}$.

b) Exhibit two functions f, g in $C(\mathbb{R})$ for which $fg \in (i)$ yet $f \notin (i)$ and $g \notin (i)$.

c) Let X be a Tychonoff space with more than one point. Prove that there are two functions $f, g \in C(X)$ such that $fg \equiv 0$ on X yet neither f nor g is identically 0 on X.

d) Prove that there are exactly two functions $f \in C(\mathbb{R})$ for which $f^2 = f$. (*Here,* $f^2(x)$ *means* $f(x) \cdot f(x)$*, not* $f(f(x))$*.*)

e) Prove that there are exactly *c* functions *f* in $C(\mathbb{Q})$ for which $f^2 = f$.

For those who know a bit of algebra: an element $f \in C(X)$ *for which* $f^2 = f$ *is called an* *idempotent* *in* $C(X)$*). Part d) shows that* $C(\mathbb{R})$ *and* $C(\mathbb{Q})$ *are not isomorphic rings since they have different numbers of idempotents. Is either isomorphic to* $C(\mathbb{N})$?

A classic part of general topology is the exploration of the relationship between the space X *and the rings* $C(X)$ *and* $C^*(X)$*. For example, if* X *and* Y *are homeomorphic, then* $C(X)$ *is isomorphic to* $C(Y)$*. This necessarily implies that* $C^*(X)$ *is isomorphic to* $C^*(Y)$ *also (why?). The question "when does isomorphism imply homeomorphism" is more difficult.*

Another important area of study is how the maximal ideals of the ring $C(X)$ *are related to the topology of* X*. The best introduction to this material is the classic* *Rings of Continuous Functions* *(Gillman-Jerison).*

f) Let $D(\mathbb{R})$ be the set of differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Are the rings $C(\mathbb{R})$ and $D(\mathbb{R})$ isomorphic? *Hint: An isomorphism between* $C(\mathbb{R})$ *and* $D(\mathbb{R})$ *preserves cube roots.*

4. A space X is called pseudocompact (*Definition IV.8.7*) if every continuous $f : X \to \mathbb{R}$ is bounded, that is, if $C(X) = C^*(X)$. Consider the following condition on a topological space X:

(*) If $V_1 \supseteq V_2 \supseteq ... \supseteq V_n \supseteq ...$ is a decreasing sequence of nonempty open sets, then $\bigcap_{n=1}^{\infty} \text{cl}\, V_n \neq \emptyset$.

a) Prove that if X satisfies (*), then X is pseudocompact.

b) Prove that if X is Tychonoff and pseudocompact, then X satisfies (*).

Note: Thus, for Tychonoff spaces, (*) gives an "internal" characterization of pseudocompactness – that is, a characterization making no explicit mention of $\mathbb{R}$*.)*

5. Let X be a Tychonoff space.

a) Suppose $F, K \subseteq X$ where F is closed, K is compact. X and $F \cap K = \emptyset$. Prove that there is an $f \in C(X)$ such that $f \mid K = 0$ and $f \mid F = 1$. (*This is another example of the rule of thumb that "compact spaces act like finite spaces." If you need to, try proving the result first with* K *a finite set.*.)

b) Suppose $p \in U$, where U is open in X. Prove $\{p\}$ is a G_δ set in X iff there exists a continuous function $f : X \to [0, 1]$ such that $f^{-1}(1) = \{p\}$ and $f|X - U = 0$.