Math 418, Spring 2010
Homework 5

Homework 5 is due in class on Thursday, March 18.

1. Let (X, d) be a metric space and $S \subseteq X$. Prove that if each continuous $f\colon S \to \mathbb{R}$ extends to a continuous $g : X \to \mathbb{R}$, then S is closed. (*The converse, of course, is from Tietze's Extension Theorem.)*

2. Prove that a Hausdorff space X is normal iff for each finite open cover $\mathcal{U} = \{U_1, \ldots , U_n\}$ of X, there exist continuous functions $f_i : X \to [0,1]$ $(i = 1, \ldots, n)$ such that $\sum_{i=1}^{n} f_i(x) = 1$ for each $x \in X$ and such that, for each i, $f_i \mid X - U_i = 0$. (*Such a set of functions is called a <u>partition of unity subordinate to the finite cover</u> $\mathcal{U}$.*)

3.. Suppose X is a T_1 space. X is called <u>perfectly normal</u> if whenever A and B are disjoint nonempty closed sets in X, there is an $f \in C(X)$ with $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

a) Prove that every metric space (X, d) is perfectly normal.

b) Prove that X is perfectly normal iff X is T_4 and every closed set in X is a G_δ-set.

4. A space X is called <u>locally compact</u> if each $x \in X$ has a neighborhood base consisting of compact neighborhoods. For example: every discrete space is locally compact, $\mathbb{R}^n$ is locally compact, and $\mathbb{Q}$ is not locally compact.

a) Prove that a compact Hausdorff space is locally compact.

Suppose X is a locally compact Hausdorff space that is not compact. For $x \in X$, let $\mathcal{B}_x$ be a neighborhood base at x consisting of compact neighborhoods. Let p be a point not in X, and define $X^* = X \cup \{p\}$. Put a topology on X^* by using the following definition for neighborhood bases:

$$\begin{cases} \mathcal{B}_x, & \text{for } x \in X \\ \mathcal{B}_p = \{B \subseteq X^* : \ p \in B \text{ and } X^* - B \text{ is compact}\} \end{cases}$$

(*You can assume that this definition satisfies the conditions in the Neighborhood Base Theorem III.5.2*).

b) Prove that X^* is a compact Hausdorff space and that X is dense in X^*.

c) Suppose X is a noncompact Hausdorff space, but <u>not</u> locally compact and we construct X^* using exactly the same definition. In that case, which statements in b) are no longer necessarily true? (*Aside: ask yourself the same questions if the definition for X^* is applied to a compact Hausdorff space X; or to a locally compact noncompact X that is not Hausdorff.*

(OVER $\longrightarrow$)

d) Suppose again that X is a locally compact, noncompact Hausdorff space, that $q \notin X$ and that $Y = X \cup \{q\}$ is a compact Hausdorff space with X as a dense subspace. Define $h : X^* \to Y$ by $h(p) = q$ and $h(z) = z$ for $z \in X$. Prove that h is a homeomorphism.

e) Suppose $X = (0, 1)$ and we construct X^*. What familiar space is X^*?
Note: the answer is the same if $X = \mathbb{R}$ since $(0, 1)$ and $\mathbb{R}$ are homeomorphic: this follows easily from the argument in d).