Math 418, Spring 2010
Homework 6

Homework 6 is due in class on Thursday, April 1

1. Suppose $(X, \leq)$ is a poset in which every nonempty subset contains a largest and smallest element. Prove that $(X, \leq)$ is a finite chain.

2. Let $(X, \leq)$ be an infinite poset. A subset C of X is called totally unordered if no two distinct elements of C are comparable, that is:

$$\forall a \in C \ \forall b \in C \quad (a \leq b) \ \Leftrightarrow \ (a = b)$$

Prove that either X has a subset C which is an infinite chain or X has a totally unordered infinite subset C.

3. Give examples of subsets of $\mathbb{Q}$ that represent the order types:

a) $\omega_0^2 + \omega_0 \cdot 3 + 2$

b) $\omega_0^3 + \omega_0$

4. Let η be the order type of $\mathbb{Q}$.

a) Give an example of a set $B \subseteq \mathbb{Q}$ such that neither B nor $\mathbb{Q} - B$ has order type η.

b) Prove that if A is a set of type η and $B \subseteq A,$ then either B or $A - B$ contains a subset of type η.

c) Prove or disprove: $\eta + 1 + \eta = \eta$.

Hint: Cantor's characterization of $\mathbb{Q}$ *as an ordered set may be helpful.*

5. Suppose C is a subset of $\mathbb{R}$ which is well-ordered (in the usual order on $\mathbb{R}$). Prove that C is countable.

(over ⟶)

6. Prove the following facts about ordinal numbers $\alpha,\ \beta,\ \gamma$:

a) if $\beta > 0$, then $\alpha + \beta > \alpha$

b) if $\alpha > \beta$, then there exists a unique γ such that $\alpha = \beta + \gamma$

We might try using b) to define subtraction of ordinals: $\alpha - \gamma = \beta$ if $\alpha = \beta + \gamma$. However this is perhaps not such a good idea. (Consider $\omega_0 = \alpha,\ \beta = 1$). Problems arise because ordinal addition is not commutative.

7. *Recall definitions and results from Homework 5; see Homework 5 Solutions.*
A compactification of a Hausdorff space X is a compact Hausdorff space Y which contains X (or a topological copy of X) as a dense subspace. In Homework 5, you proved that every locally compact noncompact Hausdorff space X has a "one-point" compactification X^* and that a one-point compactification is unique up to homeomorphism (so that we can speak of the one point compactification of X).
For example, the one point compactification of $(0,1)$ is the circle S^1 : S^1 is a compact Hausdorff space and, for any $p \in S^1$, $S^1 - \{p\}$ is a dense subspace homeomorphic to $(0,1)$.

Suppose X is a locally compact T_2 space that is separable and not compact. Show that the one-point compactification X^* is metrizable.

8. *Again, from Homework 5: some very nice spaces do not have a one-point compactification – for example $\mathbb{Q} \cap (0,1)$, because this space is non locally compact.*
And sometimes a space has other compactifications. For example, $[0,1]$ is a compactification of $\mathbb{Q} \cap (0,1)$, and, for example, $[0,1]$ is a "2-point compactification" of $(0,1)$.

Prove that a space X has a compactification iff X is a Tychonoff space.

Hints: consider the embedding theorems; you can also appeal to the Tychonoff Product Theorem (although we have not yet proved it).