

Math 418, Spring 2010
Homework 7

Homework 7 is due in class on Thursday, April 15

1. Let X be a first countable space. Suppose that for each $\alpha < \omega_1$, F_α is a closed subset of X and that $F_{\alpha_1} \subseteq F_{\alpha_2}$ whenever $\alpha_1 \leq \alpha_2 < \omega_1$. Prove that $\bigcup\{F_\alpha : \alpha < \omega_1\}$ is closed in X .

2. a) Let A and B be disjoint closed sets in $[0, \omega_1]$. Prove that at least one of A and B is compact and bounded away from ω_1 . (A set C is bounded away from ω_1 if $C \subseteq [0, \alpha]$ for some $\alpha < \omega_1$.)

b) Prove that a closed set F in $[0, \omega_1)$ is a zero set iff F either contains some “tail” $[\alpha, \omega_1)$ or F is bounded away from ω_1 .

3. A space X is called σ -compact if X can be written as a countable union of compact sets.

a) Prove $[0, \omega_1)$ is not σ -compact.

b) Use part a) (or some other method), to prove that $[0, \omega_1)$ is not Lindelöf.

4. For a metric space (X, d) , we defined (see Definition 10.3) collections \mathcal{G}_α ($\alpha < \omega_1$) and the collection of Borel sets $\mathcal{B} = \bigcup\{\mathcal{G}_\alpha : \alpha < \omega_1\}$.

a) Let \mathcal{F}_0 be the family of closed sets in X . For $\alpha < \omega_1$, define families

$$\mathcal{F}_\alpha = \begin{cases} \{\bigcap_{n=1}^{\infty} F_n : F_n \in \mathcal{F}_{\beta_n}, \beta_n < \alpha\} & \text{if } \alpha \text{ is even} \\ \{\bigcup_{n=1}^{\infty} F_n : F_n \in \mathcal{F}_{\beta_n}, \beta_n < \alpha\} & \text{if } \alpha \text{ is odd} \end{cases}$$

\mathcal{F}_1 is the collection of countable unions of closed sets (called F_σ -sets) and \mathcal{F}_2 is the family of countable intersections of F_σ sets (called $F_{\sigma\delta}$ -sets). Next comes the collection of $F_{\sigma\delta\sigma}$ sets, and so on.

Prove that $\mathcal{F}_\alpha \subseteq \mathcal{G}_\beta$ for all $\alpha < \beta < \omega_1$. (It is also true that $\mathcal{G}_\alpha \subseteq \mathcal{F}_\beta$ for $\alpha < \beta < \omega_1$: since the proof is very similar, you needn't do it here; you can assume this fact if needed.)

It follows that $\mathcal{B} = \bigcup\{\mathcal{F}_\alpha : \alpha < \omega_1\}$. We can build the Borel sets “from the bottom up” beginning with either the open sets or the closed sets. Would this be true if we defined Borel sets the same way in an arbitrary topological space?

b) Suppose X and Y are separable metric spaces. A function $f: X \rightarrow Y$ is called Borel-measurable (or B-measurable, for short) if $f^{-1}[B]$ is a Borel set in X whenever B is a Borel set in Y . Prove that f is B-measurable iff $f^{-1}[O]$ is Borel in X whenever O is open in Y .

c) Suppose X and Y are separable metric spaces. Prove that there are at most c B-measurable maps f from X to Y .

Hint: If $f^{-1}[O] = g^{-1}[O]$ for every open O , must $f = g$?

5. Prove that the continuum hypothesis ($c = \aleph_1$) is true iff \mathbb{R}^2 can be written as $A \cup B$ where A has countable intersection with every horizontal line and B has countable intersection with every vertical line.

Hints: \Rightarrow : If CH is true, \mathbb{R} can be indexed by the ordinals $< \omega_1$. See Math 417, HW 2, #8 = Exercise 1.E44. The solution is similar.

\Leftarrow : If CH is false, then $c > \aleph_1$. Suppose A meets every horizontal line only countably often and that $A \cup B = \mathbb{R}^2$. Show that B meets some vertical line uncountably often by letting Q be the union of any \aleph_1 horizontal lines and examining $\pi_X[Q \cap A]$.

6. A cover of the space X is called irreducible if it has no proper subcover.

a) Give an example of an open cover of a noncompact space which has no irreducible subcover.

b) Prove that X is compact iff every open cover has an irreducible subcover.

Hint (\Leftarrow) Let \mathcal{U} be any open cover and let \mathcal{A} be a subcover of the smallest possible cardinality m . Let γ be the least ordinal of cardinality m , and write $\mathcal{A} = \{U_\alpha : \alpha < \gamma\}$. Then consider $\{V_\beta : \beta < \gamma\}$, where $V_\beta = \bigcup \{U_\alpha : \alpha < \beta\}$.