Math 418, Spring 2010
Homework 8

Homework 8 is due in class on Thursday, April 29 (last day of class)

Not to hand in:

Two set theory students, Ray and Debra, are arguing:

> Ray: "There must be a maximal countable set of real numbers. Look: partially order the countable infinite subsets of $\mathbb{R}$ by inclusion. Now every chain of such sets has an upper bound (remember, a countable union of countable sets is countable), so Zorn's Lemma gives us a maximal element."
>
> Debra: "I don't know anything about Zorn's Lemma but it seems to me that you can always add another real to any countable set of real numbers and still have a countable set. So how can there be a largest one?"
>
> Ray: "I didn't say largest! I said maximal!"

Resolve their dispute.

To hand in:

1. A cover of the space X is called irreducible if it has no proper subcover.

a) Give an example of an open cover of a noncompact space which has no irreducible subcover.

b) Prove that X is compact iff every open cover has an irreducible subcover.

Hint ($\Leftarrow$) Let $\mathcal{U}$ be any open cover and let $\mathcal{A}$ be a subcover of the smallest possible cardinality m. Let γ be the least ordinal of cardinality m, and write $\mathcal{A} = \{U_\alpha : \alpha < \gamma\}$. Then consider $\{V_\beta : \beta < \gamma\}$, where $V_\beta = \bigcup\{U_\alpha : \alpha < \beta\}$.

2. Let $(C, \leq)$ be an uncountable chain in which each element has only countably many predecessors. Suppose $f : C \to \mathbb{R}$ and that $f(\lambda) > 0$ for each $\lambda \in C$. Show that the net f does not converge to 0 in $\mathbb{R}$.

3. Let (X, d) be a metric space and $f : [0, \omega_1) \to X$ a function given by $f(\alpha) = x_\alpha$. Show that the net (x_α) converges iff (x_α) is eventually constant.

(OVER $\longrightarrow$)

4. a) Suppose X is infinite set with the cofinite topology. Let $\mathcal{F}$ be the filter generated by the filter base consisting of all cofinite sets. To what points does $\mathcal{F}$ converge?

b) Translate the work in part a) into statements about nets.

5. a) Suppose $\mathcal{U}$ is an ultrafilter in X and that $A_1 \cup ... \cup A_n \in \mathcal{U}$. Prove that one of the sets A_i must be in $\mathcal{U}$.

b) "By duality." there is a corresponding result about universal nets. State that result and then prove it directly (*without referring to any results about filters*).

6. Let $\mathcal{U}$ be a free ultrafilter in $\mathbb{N}$ and let $\Sigma = \mathbb{N} \cup \{\sigma\}$, where $\sigma \notin \mathbb{N}$.

Define a topology $\mathcal{T}$ on Σ by $\mathcal{T} = \{O : O \subseteq \mathbb{N}$ or $O = U \cup \{\sigma\}$ where $U \in \mathcal{U}\}$

a) Prove that Σ is T_4 and that $\mathbb{N}$ is dense in Σ.

b) Prove that a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ cannot have a countable base.
Hint: $\mathcal{B} = \{U_1, ..., U_n, ...\}$ *is a base. Since* $\mathcal{U}$ *is free, each set in* $\mathcal{U}$ *must be infinite. Why?*
To prove $\mathcal{B}$ *cannot be a base for* $\mathcal{U}$*: by induction, create a set* A *such that neither* A *nor* $\mathbb{N} - A$ *contains one of the* $U_n's$.)

c) Prove that no sequence in $\mathbb{N}$ can converge to σ (and therefore there can be no countable neighborhood base at σ in Σ)
Hint: Work you did in b) might help.

Thus, Σ *is another example of a countable space where only one point is not isolated and which is not first countable, See the space* L *in Example III.9.8.*

d) How would the space Σ be different if $\mathcal{U}$ were a fixed ultrafilter?

Comment: if $\mathcal{U}'$ *is a free ultrafilter on* $\mathbb{N}$ *and* $\mathcal{U}' \neq \mathcal{U}$*, then the corresponding spaces* Σ' *and* Σ *may not be homeomorphic: the neighborhood systems of* σ *may look quite different. In this sense, free ultrafilters in* $\mathbb{N}$ *do not all "look alike."*