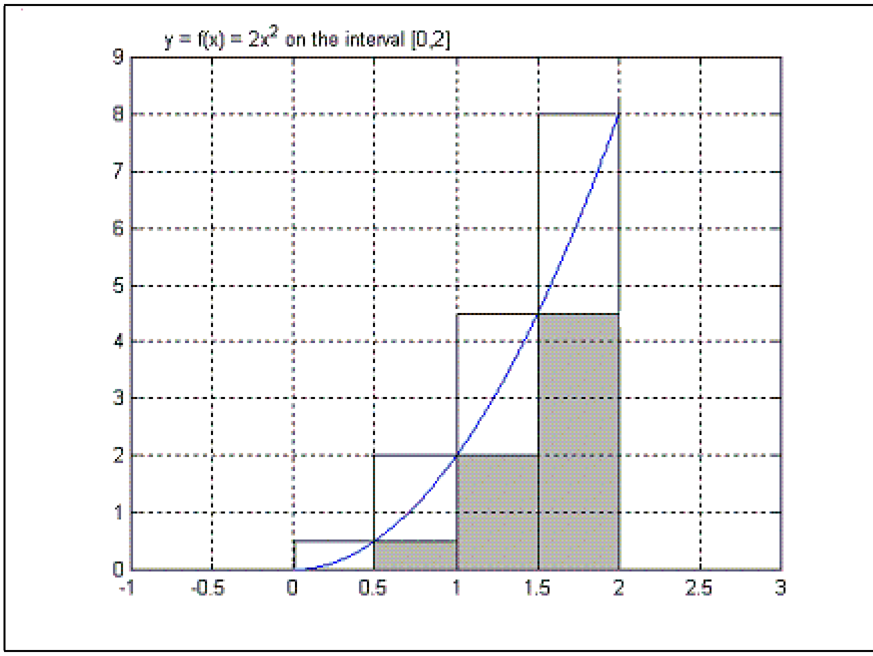
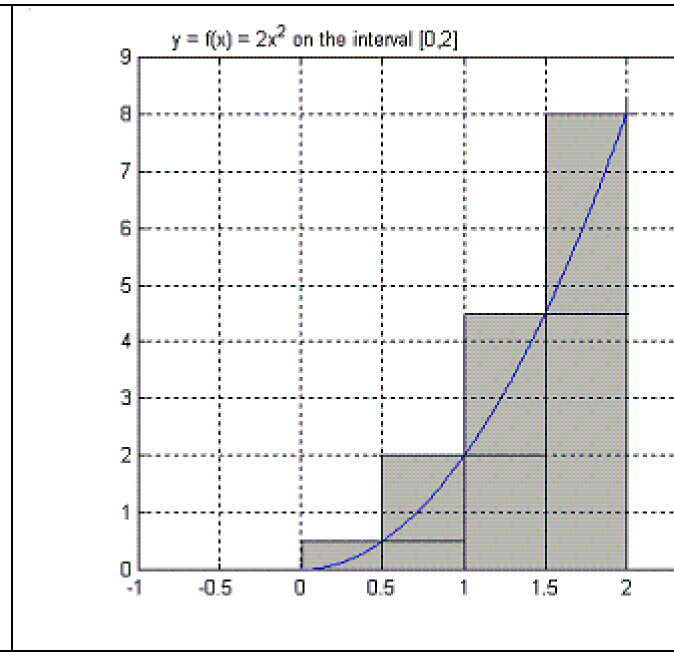
Approximating the area under a graph and above the x-axis

http://www.math.wustl.edu/~freiwald/Math131/areas.pdf



Find shaded area (it is called L_4): $L_4 =$??
($the leftmost rectangle has height $= 0$ and base from 0 to $\frac{1}{2}$)



Find the shaded area (it is called R_4): $R_4 =$??

In both pictures there are 4 rectangles: their bases are of equal length, $\Delta x = \frac{2-0}{4} = \frac{1}{2}$.
In L_4, the height of each rectangle is the distance up to the graph at the <u>left</u> endpoint. So the heights are
$f(0),\ f(\frac{1}{2}),\ f(1),\ f(\frac{3}{2})$
In R_4, the height of each rectangle is the distance up to the graph at the <u>right</u> endpoint. So the heights are
$f(\frac{1}{2}),\ f(1),\ f(\frac{3}{2}),\ f(2)$

Therefore $L_4 = \text{sum of areas} = f(0)\Delta x + f(\frac{1}{2})\Delta x + f(1)\Delta x + f(\frac{3}{2})\Delta x$
and $R_4 = \text{sum of areas} = f(\frac{1}{2})\Delta x + f(1)\Delta x + f(\frac{3}{2})\Delta x + f(2)\Delta x$

In $\sum$-notation, we could write : $R_4 = \sum_{i=1}^{4} f(0 + i\Delta x)\,\Delta x = \sum_{i=1}^{4} f(i\Delta x)\,\Delta x$

And we could write: $L_4 = \sum_{i=1}^{4} f((i-1)\Delta x)\,\Delta x$

Because f is <u>increasing,</u> L_4 <u>underestimates</u> the area under $y = 2x^2$ above $[0,2]$, and R_4 overestimates it.

If we use a larger number of subdivisions, L_4 should give a better (under)estimate of the area under the graph, and R_4 should give a better (over)estimate of the area under the graph. Turn over for a short table.

n	L_n	R_n	
4	3.5000	7.5000	
100	5.2536	5.4136	
1000	5.3253	5.3413	For $n = 1000000$, more precision actually gives
100000	5.3333	5.3334	$L_{1000000} \approx$ 5.33332533333600
1000000	5.3333	5.3333	$\mathrm{R}_{1000000} \approx$ 5.33334133333600

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \infty & ? & ? \\ & \frac{16}{3} & \frac{16}{3} \end{array}$$

look like good guesses for the limits

(area under graph)