Math 131, Fall 2004
Discussion Section 10
SOLUTIONS

1. a) Find $\lim_{x\to 0}\frac{\sin 5x}{\tan 7x}$

   b) Find $\lim_{x\to\infty}\frac{e^{1+\ln x}}{e^{x-\ln x}}$

<u>Solution:</u>

a) Since the limit is of the form "$\frac{0}{0}$" we can use L'Hôpital's Rule: $\lim_{x\to 0}\frac{\sin 5x}{\tan 7x} = \lim_{x\to 0}\frac{5\cos 5x}{7\sec^2 7x}$ $= \frac{5(1)}{7(1)} = \frac{5}{7}$.

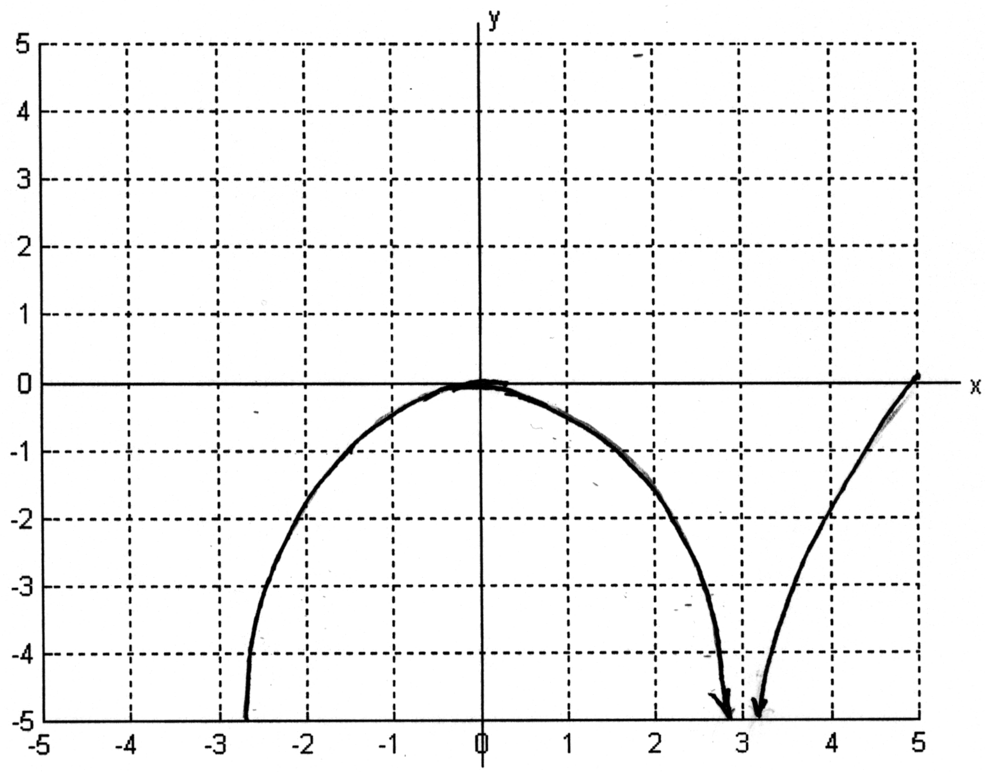
We can also do it without L'Hôpital's Rule: $\lim_{x\to 0}\frac{\sin 5x}{\tan 7x} = \lim_{x\to 0}\frac{\sin 5x}{(\sin 7x\,/\cos 7x)}$

$= \lim_{x\to 0}\frac{\sin 5x}{\sin 7x}(\cos 7x) = \lim_{x\to 0}\frac{\frac{\sin 5x}{x}}{\frac{\sin 7x}{x}}(\cos 7x)\lim_{x\to 0}\frac{5\,\frac{\sin 5x}{5x}}{\frac{7\sin 7x}{7x}}(\cos 7x) = \frac{5}{7}(1) = \frac{5}{7}$.

b) $\lim_{x\to\infty}\frac{e^{1+\ln x}}{e^{x-\ln x}} = \lim_{x\to\infty}\frac{e\cdot e^{\ln x}}{e^{x}\cdot e^{-\ln x}} = \lim_{x\to\infty}\frac{e\cdot x}{e^{x}\cdot\frac{1}{e^{\ln x}}} = \lim_{x\to\infty}\frac{e\cdot x}{\frac{e^x}{x}} = \lim_{x\to\infty}\frac{ex^2}{e^x}$
$= \lim_{x\to\infty}\frac{x^2}{e^{x-1}}$. Since this has form "$\frac{\infty}{\infty}$", we can use L'Hôpital's Rule (twice) to get $\lim_{x\to\infty}\frac{x^2}{e^{x-1}} = \lim_{x\to\infty}\frac{2x}{e^{x-1}} = \lim_{x\to\infty}\frac{2}{e^{x-1}} = 0$.

2. On the grid below, draw the graph of a function $y = f(x)$ that has all of the following properties:

$\lim_{x \to 3} f(x) = -\infty$
$f''(x) < 0$ if $x \neq 3$
$f'(0) = 0$
$f'(x) > 0$ if $x < 0$ or $x > 3$
$f'(x) < 0$ if $0 < x < 3$.



3. Let $y = f(x) = x^{5/3} - 5x^{2/3}$.

To save time, here are the first and second derivatives

$$y' = \tfrac{5}{3}x^{2/3} - \tfrac{10}{3}x^{-1/3}$$
$$y'' = \tfrac{10}{9}x^{-1/3} + \tfrac{10}{9}x^{-4/3}, \quad \text{and}$$

here are some functional values (not all of which may be relevant to the problem:

$f(-3) \approx -16.6407$	$f(-2) \approx -11.11,$	$f(-1) = -6$
$f(0) = 0$		
$f(1) = -4$	$f(2) \approx -4.76$	$f(3) \approx -4.16$

Determine the intervals where the function is increasing, decreasing, concave up or down. Find any local maxima, minima, and inflection points. Draw a reasonable sketch of the graph.

Solution: $y' = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2) = \frac{5(x-2)}{3\sqrt[3]{x}}$. The critical numbers are 2 (where $y' = 0$) and 0 (where y' does not exist).

$$\text{We have } \begin{cases} y' > 0 & \text{if } x > 2, \text{ so } f(x) \text{ is increasing there} \\ y' < 0 & \text{if } 0 < x < 2, \text{ so } f(x) \text{ is decreasing there} \\ y' > 0 & \text{if } x < 0, \text{ so } f(x) \text{ is increasing there} \end{cases}$$

Therefore we have a local max at $x = 0$ and a local min at $x = 2$.

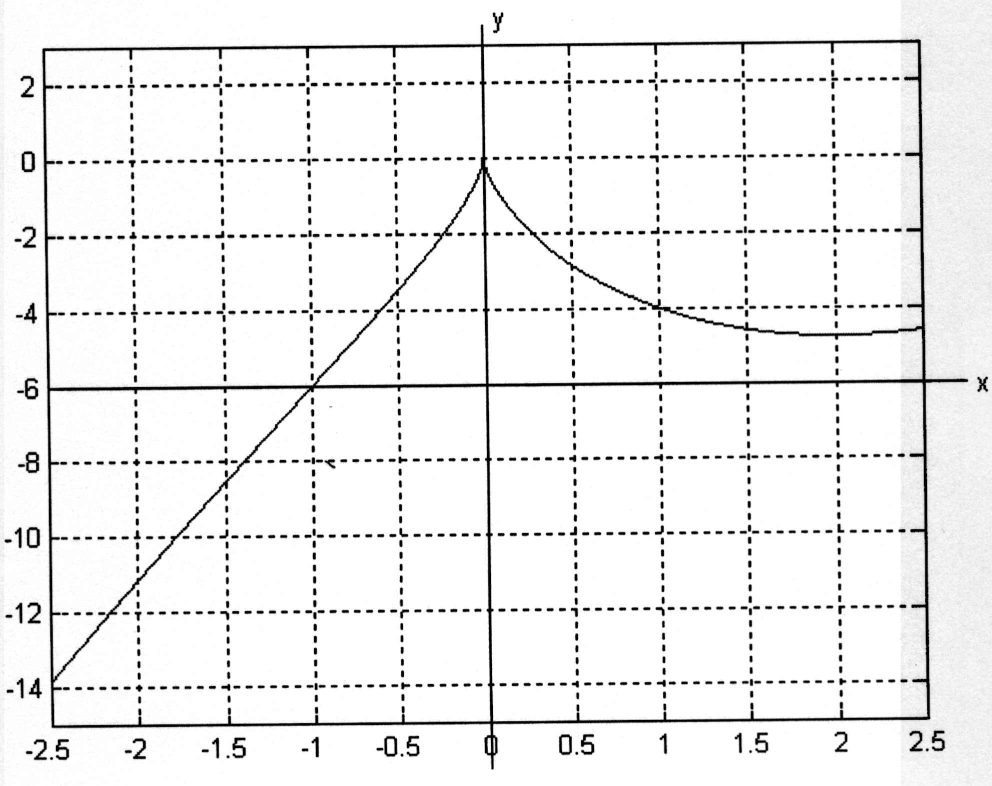
$y'' = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1) = \frac{10}{9\sqrt[3]{x^4}}(x+1)$

$y'' = 0$ if $x = -1$ and y'' does not exist if $x = 0$. These are the candidates for inflection points.

$$\text{We have } \begin{cases} y'' < 0 & \text{if } x < -1, \text{ so } f(x) \text{ is concave down there} \\ y'' > 0 & \text{if } -1 < x < 0, \text{ so } f(x) \text{ is concave up there} \\ y'' > 0 & \text{if } 0 < x, \text{ so } f(x) \text{ is also concave up there} \end{cases}$$

The only inflection point is at $x = -1$.

In the picture, notice the "sharp corner" at $x = 0$, indicating "no derivative" there. Also, the scale on the computer-drawn graph makes it hard to see the switch from concave down to concave up at $x = -1$: it's just ever so faintly visible and you'd probably overlook it if you were relying on a computer or calculator drawing to visually locate the inflection points.



y
x
2
0
-2
-4
-6
-8
-10
-12
-14
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5