Math 131, Fall 2004
Discussion Section 11 Solutions

1. Find $\lim_{x\to 0^+} (\sin x)^{\tan x}$

Solution: Let $y = (\sin x)^{\tan x}$, so $\ln y = \ln((\sin x)^{\tan x}) = (\tan x)\ln(\sin x)$
$= \frac{\ln(\sin x)}{\cot x}$.

$\lim_{x\to 0^+} \frac{\ln(\sin x)}{\cot x}$ is of the form "$\frac{-\infty}{\infty}$" so we can use L'Hôpital's Rule.

$\lim_{x\to 0^+} \frac{\ln(\sin x)}{\cot x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to 0^+} -\frac{\cos x}{\sin x}\cdot\frac{\sin^2 x}{1}$
$= \lim_{x\to 0^+} -(\cos x)(\sin x) = 0.$

Since $\ln y \to 0$ as $x \to 0^+$, we get $y = e^{\ln y} \to e^0 = 1$.

2. Find a function $f(x)$ for which $f'''(x) = 2x + 1$, $f''(0) = 0$, $f(0) = 1$ and $f(1) = 0$

Solution: $f'''(x) = 2x + 1$, so $f''(x) = x^2 + x + C$, where C is a constant.
Since $0 = f''(0) = 0 + 0 + C$, we conclude that $C = 0$ and $f''(x) = x^2 + x$.

Then $f'(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + D$, where D is a constant, so
$f(x) = \frac{1}{12}x^4 + \frac{1}{6}x^3 + Dx + E$ where E is a constant.

Since $1 = f(0) = 0 + 0 + 0 + E$, we conclude that $E = 1$ so
$f(x) = \frac{1}{12}x^4 + \frac{1}{6}x^3 + Dx + 1$.

Since $0 = f(1) = \frac{1}{12} + \frac{1}{6} + D + 1$, we have $D = -1 - \frac{1}{6} - \frac{1}{12} = -\frac{15}{12}$
so $f(x) = \frac{1}{12}x^4 + \frac{1}{6}x^3 - \frac{15}{12}x + 1$

.

3. What are the dimensions of an aluminum can that holds 40 cm^3 of juice and that uses the least material (aluminum)? *Assume the can is cylindrical and is capped on both ends.*

Solution:

Suppose the can has radius r and height h (cm). The material needed is

the surface area of the cylinder $= 2\pi rh$ (why?)
$+$ the area of the circular top of the can $= \pi r^2$
$+$ the area of the circular bottom of the can $= \pi r^2$

So we want to minimize $S = 2\pi rh + 2\pi r^2 = 2\pi(rh + r^2)$

Since $V = \pi r^2 h = 40$, we have $h = \frac{40}{\pi r^2}$, so $S = 2\pi(r \cdot \frac{40}{\pi r^2} + r^2)$
$= 2\pi(\frac{40}{\pi}\frac{1}{r} + r^2)$.

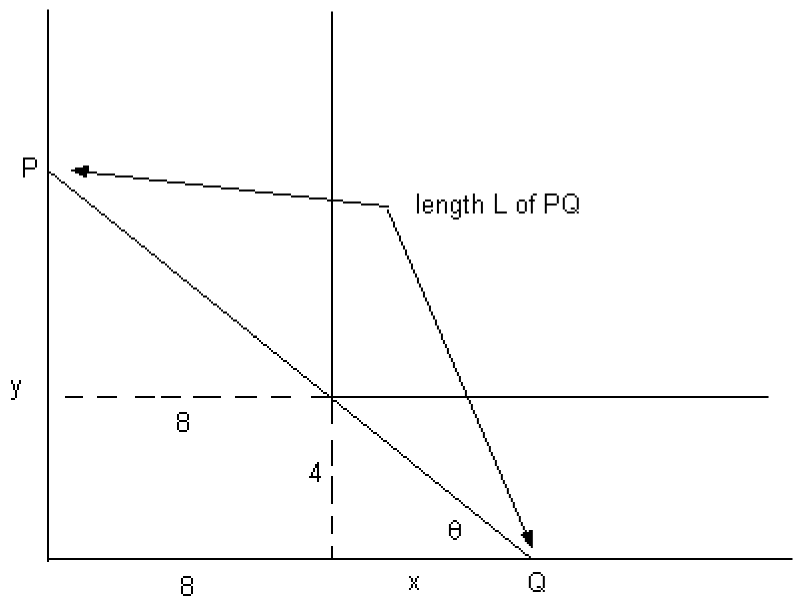
Clearly we must have $r > 0$ and, in theory anyway, r could be as large as we like.

So we need to minimize $S = 2\pi(r^2 + \frac{40}{\pi}\frac{1}{r})$, where $0 < r < \infty$.

$S' = 2\pi\,(2r - \frac{40}{\pi r^2})$. S' exists for all $r > 0$ so the only critical numbers are where $S' = 0$, that is $2r = \frac{40}{\pi r^2}$ so that $r = (\frac{20}{\pi})^{1/3}$.

It's easy to check that $S' > 0$ for $r > (\frac{20}{\pi})^{1/3}$ and $S' < 0$ if $r < (\frac{20}{\pi})^{1/3}$ so S has an absolute minimum at $r = (\frac{20}{\pi})^{1/3}$. For this r, the value of $h = \frac{40}{\pi(\frac{20}{\pi})^{2/3}} = 2(\frac{20}{\pi})^{1/3}$.
In other words, the amount of material need is smallest when the height h is twice the radius r.
(The actual values are $r \approx 1.85$ and $h \approx 3.70$).

4. One hallway which is 4 ft wide meets another hallway which is 8 ft wide in a right angle. What is the length of the longest ladder that can be carried horizontally around the corner?



(*This problem is exactly the same mathematically as the WebWork problem with the ladder going over a fence to lean against a building.*)

As the ladder is moved around the corridor, it will get stuck if it is ever longer than the line segment PQ. Therefore the longest ladder that will go around the corner is the length of the shortest line segment PQ. We want to minimize the length $L = PQ$.

The Pythagorean Theorem gives $L^2 = (8+x)^2 + y^2$
By similar triangles, $\frac{y}{8+x} = \frac{4}{x}$, so $y = \frac{4(8+x)}{x}$ and
$L^2 = (8+x)^2 + \frac{16(8+x)^2}{x^2} = (8+x)^2(1+\frac{16}{x^2})$. Therefore
$L = \sqrt{(8+x)^2(1+\frac{16}{x^2})}$ where $x > 0$.

We want to minimize L. It makes our job easier, however, to notice that L will be smallest when $(8+x)^2(1+\frac{16}{x^2})$ is smallest, so we just minimize $(8+x)^2(1+\frac{16}{x^2})$ instead and avoid needing to work with the square root.

The derivative of $(8+x)^2(1+\frac{16}{x^2})$ is

$(8+x)^2(\frac{-32}{x^3}) + 2(8+x)(1+\frac{16}{x^2}) = (8+x)[\,(8+x)(\frac{-32}{x^3}) + 2(1+\frac{16}{x^2})$
$= (8+x)(\frac{-256}{x^3} - \frac{32}{x^2} + 2 + \frac{32}{x^2}) = (8+x)(\frac{-256}{x^3} + 2).$

Since $x > 0,$ there are points where the derivative doesn't exist and the only critical numbers will be where the derivative $(8+x)(\frac{-256}{x^3} + 2) = 0.$ This happens for $x = -8$ (outside the domain in this problem) and for $x = (128)^{1/3}$.

It's easy to check that the derivative is < 0 for $x < (128)^{1/3}$ and the derivative is > 0 for $x > (128)^{1/3}$. Therefore $x = (128)^{1/3}$ gives the minimum value for $(8+x)^2(1+\frac{16}{x^2})$. The minimum length is $L = \sqrt{(8+128^{1/3})^2 \cdot (1+\frac{16}{(128)^{2/3}})}$ ≈ 16.65 ft. and this is the length of the longest ladder that will go around the corner,

Here is a different way to set up the same problem. It is easier (less "algebra-intensive") if you're comfortable with the trig functions. A few details are omitted.

In the large triangle, $\frac{L}{8+x} = \sec\theta$, so $L = (8+x)\sec\theta$
In the small triangle, $\frac{4}{x} = \tan\theta$, so $x = 4\cot\theta$.
Therefore $L = (8+4\cot\theta)\sec\theta = 8\sec\theta + 4\csc\theta$ (where $0 < \theta < \frac{\pi}{2}$).
We want to minimize the length L. We set

$$\begin{aligned} L' = \;& 8\sec\theta\tan\theta - 4\csc\theta\cot\theta = 0 \\ & 2\sec\theta\tan\theta = \csc\theta\cot\theta \\ & \frac{2}{\cos\theta}\frac{\sin\theta}{\cos\theta} = \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta} \\ & 2\sin^3\theta = \cos^3\theta \\ & \tan^3\theta = \frac{1}{2} \\ & \tan\theta = \frac{1}{2^{1/3}}, \text{ so } \theta = \arctan(\frac{1}{2^{1/3}}) \end{aligned}$$

Substituting this value of θ into $L = 8\sec\theta + 4\csc\theta$
gives $L \approx 16.65$ ft.