Math 131, Fall 2004
Discussion Section 12 Solutions

1. A point is moving along a straight line. Its velocity $v(t)$ is $\begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 \leq t \leq 3 \end{cases}$ where v is measured in m/sec. Its position at time 0 is $s(0) = 2$.

a) Use antiderivatives to get a formula for the position function $s(t)$.

Solution: $s(t) = \begin{cases} \frac{t^2}{2} + C & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} + D & 1 \leq t \leq 3 \end{cases}$. The initial condition $s(0) = 2$ gives us that $C = 2$, so $s(t) = \begin{cases} \frac{t^2}{2} + 2 & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} + D & 1 \leq t \leq 3 \end{cases}$. The first line of the formula gives us that $s(1) = \frac{1}{2} + 2 = \frac{5}{2}$. The second line gives us that $s(1) = 2 - \frac{1}{2} + D$. These values for $s(1)$ must agree so $\frac{5}{2} = \frac{3}{2} + D$ and $D = 1$. Therefore

$$s(t) = \begin{cases} \frac{t^2}{2} + 2 & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} + 1 & 1 \leq t \leq 3 \end{cases}$$

b) Find $s(3) - s(0)$ ($=$ the net displacement of the point as t goes from 0 to 3). What are the units?

Solution: $s(3) - s(0) = \frac{5}{2} - 2 = \frac{1}{2}$ meters.

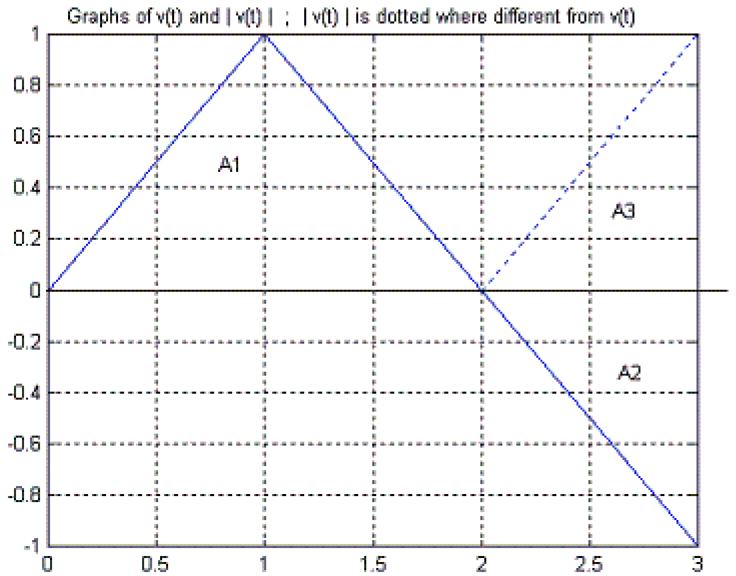
c) What is the total distance the point traveled between times 0 and 3?

Solution: For $0 \leq t \leq 2: \ v \geq 0$, so the point is moving in the positive direction and the displacement $=$ distance traveled is $s(2) - s(0) = 3 - 2 = 1$ meter.

For $2 \leq t \leq 3$: $v(t) < 0$, so the point is moving in the negative direction and the displacement $= s(3) - s(2) = \frac{5}{2} - 3 = -\frac{1}{2}$; the distance traveled is $|s(3) - s(2)| = \frac{1}{2}$.

The total distance traveled is $s(2) - s(0) + |s(3) - s(2)| = \frac{3}{2}$ meters.

d) Draw the velocity function $v(t)$ on the grid below.
e) By computing areas, evaluate $\int_0^3 v(t)\,dt$. What are its units?
f) Add the graph of $|v(t)|$ to the grid, and find the value of $\int_0^3 |v(t)|\,dt$. What are its units and what does it represent?



<u>Solution</u>: The graphs for parts d) and f) are indicated above.
Label the three triangular areas A_1, A_2 and A_3 as shown.

For e): $\int_0^3 v(t)\,dt = A_1 - A_2 = \frac{1}{2}(2)(1) - \frac{1}{2}(1)(1) = \frac{1}{2}$ meters.
This represents the <u>net displacement</u> of the moving point between times 0 and 3.

For f) $\int_0^3 |v(t)|\,dt = A_1 + A_3 = \frac{1}{2}(2)(1) + \frac{1}{2}(1)(1) = \frac{3}{2}$ meters.
Geometrically, $\int_0^3 |v(t)|\,dt$ is the area under the pictured graph of $|v(t)|$. Physically, it represents the <u>total distance traveled</u> by the point for $0 \leq t \leq 3$. This is different because, when $v(t)$ is negative, some positive and negative distances traveled cancel out. in computing $\int_0^3 |v(t)|$, sign differences on v are ignored and no cancellations of distances traveled occur. By the way, $|v(t)|$ is called the <u>speed</u> of the moving point: it is different from velocity in that it ignores sign.

2. a) If $\int_0^1 f(t)\,dt = 2$, $\int_0^4 f(t)\,dt = -6$, and $\int_3^4 f(t)\,dt = 1$, what is $\int_1^3 2f(t)\,dt$?

<u>Solution</u>: $\int_0^1 f(t)\,dt + \int_1^3 f(t)\,dt + \int_3^4 f(t)\,dt = \int_0^4 f(t)\,dt$, so

$$2 + \int_1^3 f(t)\,dt + 1 = -6, \text{ so}$$

$$\int_1^3 f(t)\,dt = -9, \text{ so}$$

$$\int_1^3 2f(t)\,dt = -18$$

b) In class, we used the definition of an integral (a limit of Riemann sums) to calculate that $\int_0^2 2x^2\,dx = \frac{16}{3}$. What then is $\int_0^2 5 - 6x^2\,dx$?

Solution: $\int_0^2 5 - 6x^2\,dx = \int_0^2 5\,dx - 3\int_0^2 2x^2\,dx = 10 - 3(\frac{16}{3}) = -6.$

c) Suppose a function $f(x)$ satisfies $4 \leq f(x) \leq 9$ for $-3 \leq x \leq 0$. What can you say about the size of $\int_{-3}^0 f(x)\,dx$?

Solution: (See property 8, p. 366 in text) $4 \leq f(x) \leq 9$, so $4(0-(-3)) \leq \int_{-3}^0 f(x)\,dx \leq 9(0-(-3))$, that is, $12 \leq \int_{-3}^0 f(x)\,dx \leq 27$.

d) For the function $f(x)$ in part c), what can you say about $\int_{-1}^0 f(x)\,dx - \int_1^0 f(x)\,dx + \int_1^{-1} f(x)\,dx$?

Solution: $\int_{-1}^0 f(x)\,dx - \int_1^0 f(x)\,dx + \int_1^{-1} f(x)\,dx$

$= \int_{-1}^0 f(x)\,dx + \int_0^1 f(x)\,dx - \int_{-1}^1 f(x)\,dx$

$= \int_{-1}^1 f(x)\,dx - \int_{-1}^1 f(x)\,dx = 0.$

e) Evaluate $\int_1^4 \frac{1}{x} - \sqrt{x}\,dx$

Solution: An antiderivative for $\frac{1}{x} - \sqrt{x}$ is $F(x) = \ln|x| - \frac{2}{3}x^{3/2}$.
By the Evaluation Theorem, $\int_1^4 \frac{1}{x} - \sqrt{x}\,dx = F(4) - F(1)$
$= (\ln 4 - \frac{2}{3}(8)) - (0 - \frac{2}{3}) = \ln 4 - \frac{14}{3}$