1. a) In the figure below, find the area of the white region.

![Diagram of a region with a white area and a gray area, with a curve $y = \frac{1}{3}x^2$ and the range $0 \leq x \leq 1$.

**Solution:** The area of the white region is $\int_0^1 3x^2 \, dx - x^3 \bigg|_0^1 = 1^3 - 0^3 = 1$.

b) With using any additional calculus, find the area of the gray region.

**Solution:** The area of the gray region is the area of the whole rectangle minus the area of the white region $= (3)(1) - 1 = 2$.

c) Write two different integrals you could use to determine the area of the gray region.

**Solution:**

1) The gray area is the rectangular area under the graph of $y = 3$ over $[0, 1]$ (area of the white region)

$$= \int_0^1 3 \, dx - \int_0^1 3x^2 \, dx = \int_0^1 3 - 3x^2 \, dx$$

$$= (3x - x^3) \bigg|_0^1 = (3 - 1) - (0 - 0) = 2$$

2) Since $y = 3x^2 \ (0 \leq x \leq 1)$, we can write $x = \left(\frac{y}{3}\right)^{1/2}$, $0 \leq y \leq 3$. The gray area is the area “under” ( = “to the left of”) the graph of $\left(\frac{y}{3}\right)^{1/2}$ for $0 \leq y \leq 3$. (Turn your head to the right and look at the picture sideways, so that the $y$-axis is “horizontal” with positive direction left). The gray area is $\int_0^3 \sqrt{\frac{y}{3}} \, dy$.

(Since $\int \left(\frac{y}{3}\right)^{1/2} \, dy = 2\left(\frac{y}{3}\right)^{3/2} + C$ (check by differentiating!!), we can calculate
\[ \int_0^3 \left( \frac{y}{3} \right)^{1/2} dy = 2 \left( \frac{y}{3} \right)^{3/2} \bigg|_0^3 = 2(1)^{3/2} - 2(0)^{3/2} = 2. \]

2. Suppose \( f(x) \) is a continuous function.

a) If \( f(-x) = -f(x) \), what can you say about the graph of \( f(x) \)? What can you say about \( \int_{-a}^{a} f(x) \, dx \)?

**Solution:** \( f(x) \) is an odd function so its graph is symmetric with respect to the origin. For example:

\[
\text{By symmetry, the shaded regions have the same area, and the integral counts the area below the } x\text{-axis as negative, so}
\int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx = -A + A = 0.
\]

b) If \( f(-x) = f(x) \), what can you say about the graph of \( f(x) \)? What can you say about \( \int_{-a}^{a} f(x) \, dx \)?

**Solution:** \( f(x) \) is an even function so its graph is symmetric with respect to the \( y\)-axis. For example:
By symmetry, the shaded areas are equal, so
\[ \int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx = A + A = 2 \int_{0}^{a} f(x) \, dx. \]

c) Find the value of \( \int_{0}^{\pi} \cos \left( \frac{1}{2} x \right) \, dx \).

**Solution:** \( \int \cos \left( \frac{1}{2} x \right) \, dx = 2 \sin \left( \frac{1}{2} x \right) + C \) (check by differentiating!!). Therefore \( \int_{0}^{\pi} \cos \left( \frac{1}{2} x \right) \, dx = 2 \sin \left( \frac{1}{2} \pi \right) \bigg|_{0}^{\pi} = 2 \sin \left( \frac{\pi}{2} \right) - 2 \sin(0) = 2. \)

d) Find the value of \( \int_{-\pi}^{\pi} x \sqrt{7x^4 + 9x^2 + 13} - 9 \cos \left( \frac{1}{2} x \right) \, dx \)

**Solution:** \( x \sqrt{7x^4 + 9x^2 + 13} \) is an odd function, and \( 9 \cos \left( \frac{1}{2} x \right) \) is an even function so (using parts a), b), and c) ), we get
\[
\int_{-\pi}^{\pi} x \sqrt{7x^4 + 9x^2 + 13} - 9 \cos \left( \frac{1}{2} x \right) \, dx \\
= \int_{-\pi}^{\pi} x \sqrt{7x^4 + 9x^2 + 13} \, dx - 9 \int_{-\pi}^{\pi} \cos \left( \frac{1}{2} x \right) \, dx \\
= 0 - 9 \int_{-\pi}^{\pi} \cos \left( \frac{1}{2} x \right) \, dx \\
= -9(2) \int_{0}^{\pi} \cos \left( \frac{1}{2} x \right) \, dx = -9(2)(2) = -36. \]

3) a) What is \( \frac{d}{ds} \int_{1}^{s} s^2 \, ds \)?

**Solution:** \( \int_{1}^{s} s^2 \, ds \) is a constant \( = \frac{7}{3} \) if you work it out, and \( \frac{d}{ds} \) (constant) = 0.

b) What is \( \frac{d}{ds} \int_{1}^{s} t^2 \, dt \)?
Solution: By the Fundamental Theorem of Calculus (Part I),
\[ \frac{d}{ds} \int_1^s t^2 \, dt = s^2 \] (In this case, the integral is so easy that you could simplify it first and avoid the Fundamental Theorem: \[ \frac{d}{ds} \int_1^s t^2 \, dt = \frac{d}{ds} \left( \frac{s^3}{3} - \frac{1}{3} \right) = s^2. \])

c) What is \( \frac{d}{dt} \int_1^s t^2 \, dt \)?

Solution: \( \int_1^s t^2 \, dt \) is a function of \( s \), not \( t \): so \( \frac{d}{dt} \int_1^s t^2 \, dt = 0. \) (Actually, this assumes that \( s \) and \( t \) are independent — that \( s \) is not really some function of \( t \). To allow for that possibility and still be right, we could use the chain rule: \( \frac{d}{dt} \int_1^s t^2 \, dt = s^2 \cdot \frac{ds}{dt}. \) Then, if \( s \) is really independent of \( t \), we have \( \frac{ds}{dt} = 0 \) so \( \frac{d}{dt} \int_1^s t^2 \, dt = 0 \) as we had before.)

d) What is \( \frac{d}{dz} \int_0^z \sqrt{t^2 + t + 1} \, dt \)?

Solution: Since both limits on the integral are the same, \( \int_0^z \sqrt{t^2 + t + 1} \, dt = \frac{d}{dz} (0) = 0. \) (Of course, you could work it out the "longer way":
\[ \frac{d}{dz} \int_0^z \sqrt{t^2 + t + 1} \, dt = \frac{d}{dz} \int_0^0 \sqrt{t^2 + t + 1} \, dt + \frac{d}{dz} \int_0^z \sqrt{t^2 + t + 1} \, dt = -\frac{d}{dz} \int_0^0 \sqrt{t^2 + t + 1} \, dt + \frac{d}{dz} \int_0^z \sqrt{t^2 + t + 1} \, dt = 0. \])

e) What is \( \frac{d}{du} \int_u^{2u} 2t \, dt \)?

Solution: \( \frac{d}{du} \int_u^{2u} 2t \, dt = \frac{d}{du} \int_0^0 2t \, dt + \frac{d}{du} \int_0^{2u} 2t \, dt = \frac{d}{du} \int_0^0 2t \, dt - \frac{d}{du} \int_0^u 2t \, dt = \frac{d}{du} \int_0^u 2t \, dt - 2u = 2(2u) \frac{d}{du} (2u) - 2u = 8u - 2u = 6u. \)

f) What is \( \frac{d}{ds} \int_1^{s^2} t^2 \, dt \)?

Solution: Let \( y = \int_1^{s^2} t^2 \, dt = \int_1^u t \, dt \), where \( u = \int_1^s t^2 \, dt \). Then \( \frac{du}{ds} = s^2 \) (using the Fundamental Theorem, Part I) so
\[ \frac{dy}{ds} = \frac{du}{ds} = u \cdot \frac{du}{ds} = u \cdot s^2 = s^2 \int_1^s t^2 \, dt \]
(If we simplify further, \( s^2 \int_1^s t^2 \, dt = s^2 \cdot \frac{1}{3} \frac{d}{ds} \left( \frac{s^3}{3} \right) = s^2 \left( \frac{s^3}{3} - \frac{1}{3} \right) = \frac{s^5 - s^3}{3} \).)