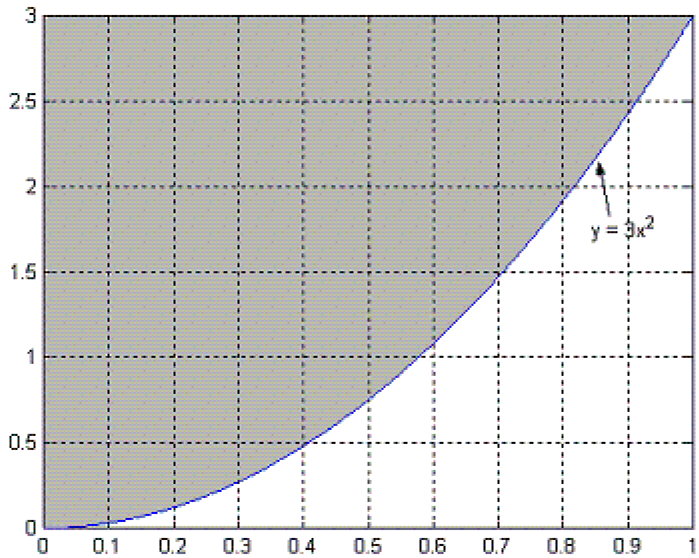
Math 131, Fall 2004
Discussion Section 13

1. a) In the figure below, find the area of the white region.



Solution: The area of the white region is $\int_0^1 3x^2\,dx = x^3|_0^1 = 1^3 - 0^3 = 1$.

b) With using any additional calculus, find the area of the gray region.

Solution: The area of the gray region is the area of the whole rectangle minus the area of the white region $= (3)(1) - 1 = 2$.

c) Write two different integrals you could use to determine the area of the gray region.

Solution:

1) The gray area is the
rectangular area under the graph of $y = 3$ over $[0,1]$ – (area of the white region)
$= \int_0^1 3\,dx - \int_0^1 3x^2\,dx = \int_0^1 3 - 3x^2\,dx$
($= (3x - x^3)|_0^1 = (3-1) - (0-0) = 2$)

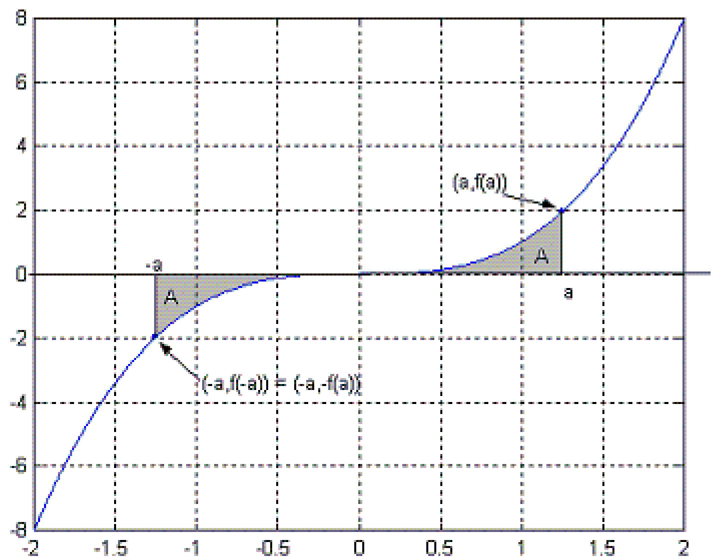
2) Since $y = 3x^2$ $(0 \le x \le 1)$, we can write $x = (\frac{y}{3})^{1/2}$, $0 \le y \le 3$. The gray area is the area "under" (= "to the left of") the graph of $(\frac{y}{3})^{1/2}$ for $0 \le y \le 3$. (Turn your head to the right and look at the picture sideways, so that the y-axis is "horizontal" with positive direction left). The gray area is $\int_0^3 \sqrt{\frac{y}{3}}\,dy$.
(Since $\int (\frac{y}{3})^{1/2} dy = 2(\frac{y}{3})^{3/2} + C$ (check by differentiating!!), we can calculate

$\int_0^3 (\frac{y}{3})^{1/2}\, dy = 2(\frac{y}{3})^{3/2}|_0^3 = 2(1)^{3/2} - 2(0)^{3/2} = 2.)$

2. Suppose $f(x)$ is a continuous function.

a) If $f(-x) = -f(x)$, what can you say about the graph of $f(x)$? What can you say about $\int_{-a}^{a} f(x)\, dx$?

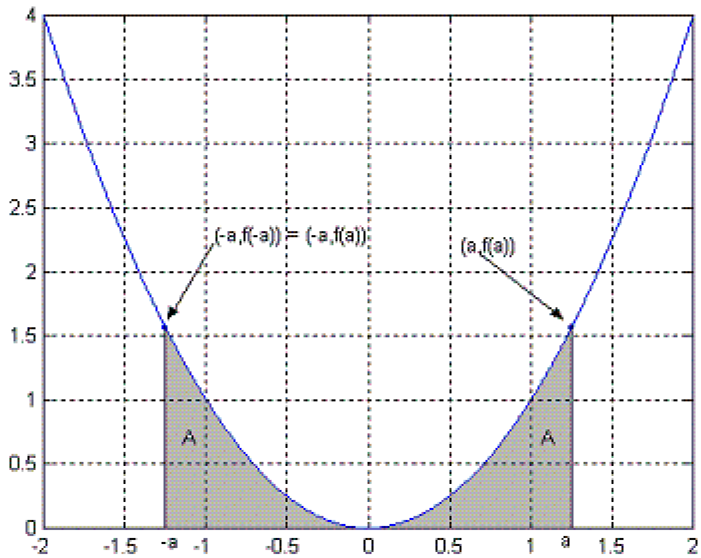
Solution: $f(x)$ is an odd function so its graph is symmetric with respect to the origin. For example:



By symmetry, the shaded regions have the same area, and the integral counts the area below the x-axis as negative, so
$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx = -A + A = 0.$

b) If $f(-x) = f(x)$, what can you say about the graph of $f(x)$? What can you say about $\int_{-a}^{a} f(x)\, dx$?

Solution: $f(x)$ is an even function so its graph is symmetric with respect to the y-axis. For example:



By symmetry, the shaded areas are equal, so
$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = A + A = 2\int_{0}^{a} f(x)\,dx.$

c) Find the value of $\int_0^{\pi} \cos\left(\frac{1}{2}x\right) dx$.

<u>Solution</u>: $\int \cos(\frac{1}{2}x)\,dx = 2\sin(\frac{1}{2}x) + C$ (check by differentiating!!).
Therefore $\int_0^{\pi} \cos\left(\frac{1}{2}x\right) dx = 2\sin(\frac{1}{2}x)|_0^{\pi} = 2\sin(\frac{\pi}{2}) - 2\sin(0) = 2.$

d) Find the value of $\int_{-\pi}^{\pi} x\sqrt[4]{7x^4 + 9x^2 + 13} \; - 9\cos(\frac{1}{2}x)\,dx$

<u>Solution</u>: $x\sqrt[4]{7x^4 + 9x^2 + 13}$ is an odd function, and $9\cos(\frac{1}{2}x)$ is an even function so (using parts a), b), and c)), we get
$\int_{-\pi}^{\pi} x\sqrt[4]{7x^4 + 9x^2 + 13} \; - 9\cos(\frac{1}{2}x)\,dx$
$= \int_{-\pi}^{\pi} x\sqrt[4]{7x^4 + 9x^2 + 13} \; - \int_{-\pi}^{\pi} 9\cos(\frac{1}{2}x)\,dx$
$= 0 - 9\int_{-\pi}^{\pi} \cos(\frac{1}{2}x)\,dx$
$= \; - 9(2)\int_0^{\pi} \cos(\frac{1}{2}x)\,dx = \; - 9(2)(2) = \; - 36.$

3) a) What is $\frac{d}{ds}\int_1^2 s^2\,ds$?

<u>Solution</u>: $\int_1^2 s^2\,ds$ is a <u>constant</u> $(= \frac{7}{3}$, if you work it out$)$, and $\frac{d}{ds}(\text{constant}) = 0.$

b) What is $\frac{d}{ds}\int_1^s t^2\,dt$?

Solution: By the Fundamental Theorem of Calculus (Part I),

$\frac{d}{ds}\int_1^s t^2\,dt \ = s^2$ (In this case, the integral is so easy that you could simplify it first and avoid the Fundamental Theorem: $\frac{d}{ds}\int_1^s t^2\,dt \ = \frac{d}{ds}(\frac{s^3}{3} - \frac{1}{3}) = s^2$.)

c) What is $\frac{d}{dt}\int_1^s t^2\,dt$?

Solution: $\int_1^s t^2\,dt$ is a function of s, not t: so $\frac{d}{dt}\int_1^s t^2\,dt = 0$.
(Actually, this assumes that s and t are independent – that s is not really some function of t. To allow for that possibility and still be right, we could use the chain rule:
$\frac{d}{dt}\int_1^s t^2\,dt = s^2 \cdot \frac{ds}{dt}$. Then, if s is really independent of t, we have $\frac{ds}{dt} = 0$ so
$\frac{d}{dt}\int_1^s t^2\,dt = 0$ as we had before.)

d) What is $\frac{d}{dz}\int_{z^3}^{z^3} \sqrt[4]{t^2+t+1}\,dt$?

Solution: Since both limits on the integral are the same, $\int_{z^3}^{z^3} \sqrt[4]{t^2+t+1} = 0$, so
$\frac{d}{dz}\int_{z^3}^{z^3} \sqrt[4]{t^2+t+1}\,dt \ = \frac{d}{dz}(0) = 0$.
(Of course, you could work it out the "longer way":
$\frac{d}{dz}\int_{z^3}^{z^3} \sqrt[4]{t^2+t+1}\,dt \ = \frac{d}{dz}\int_{z^3}^{0} \sqrt[4]{t^2+t+1}\,dt + \frac{d}{dz}\int_0^{z^3} \sqrt[4]{t^2+t+1}\,dt$
$= \ - \frac{d}{dz}\int_0^{z^3} \sqrt[4]{t^2+t+1}\,dt + \frac{d}{dz}\int_0^{z^3} \sqrt[4]{t^2+t+1}\,dt = 0$.)

e) What is $\frac{d}{du}\int_u^{2u} 2t\,dt$?

Solution: $\frac{d}{du}\int_u^{2u} 2t\,dt = \frac{d}{du}\int_u^0 2t\,dt + \frac{d}{du}\int_0^{2u} 2t\,dt$
$= \frac{d}{du}\int_0^{2u} 2t\,dt - \frac{d}{du}\int_0^u 2t\,dt = \frac{d}{du}\int_0^{2u} 2t\,dt - 2u$
$= 2(2u)\frac{d}{du}(2u) - 2u = 8u - 2u = 6u$.

f) What is $\frac{d}{ds}\int_1^{\int_1^s t^2\,dt} t\,dt$?

Solution: Let $y = \int_1^{\int_1^s t^2\,dt} t\,dt = \int_1^u t\,dt$, where $u = \int_1^s t^2\,dt$.
Then $\frac{du}{ds} = s^2$ (using the Fundamental Theorem, Part I)) so
$\frac{dy}{ds} = \frac{dy}{du}\frac{du}{ds} = u \cdot \frac{du}{ds} = u \cdot s^2 = s^2\int_1^s t^2\,dt$
(If we simplify further, $s^2\int_1^s t^2\,dt = s^2 \cdot \frac{t^3}{3}\Big|_1^s$
$= s^2(\frac{s^3}{3} - \frac{1}{3}) = \frac{s^5 - s^2}{3}$.)