Math 131, Fall 2004
Discussion Section 1 Solutions

You should pick up a copy of the Discussion Section sheet each week when you come in. If there are a few minutes before the quiz starts, you should try to work on one of the problems.

1. Some T/F questions on precalculus: if the statement is false, provide a counterexample. (*A counterexample is a specific example that shows that a statement is false. For example, we can show that the statement*

"For every real number a, there is a real number b such that $b^2 = a$"

is false by looking at a counterexample: for example, $a = -1$.)

a) For every a, b, $(a+b)^2 = a^2 + b^2$
False: for example, $(1+2)^2 = 9 \neq 5 = (1^2 + 2^2)$.
In fact, $(a+b)^2 = a^2 + 2ab + b^2$

b) For every a, b, $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$
False: for example, $\frac{1}{2} + \frac{1}{3} = \frac{5}{6} \neq \frac{1}{2+3}$
In fact, $\frac{1}{a} + \frac{1}{b} = \frac{b+a}{ab}$

c) For every $a, b \geq 0$ $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$
False: for example $5 = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 7$

d) For every a, $\sqrt{a^2} = a$
False: for example, let $a = -2$. $\sqrt{(-2)^2} = \sqrt{4} \neq -2$.
Remember, the standard convention is that $\sqrt{x}$ means the nonnegative square root of x.

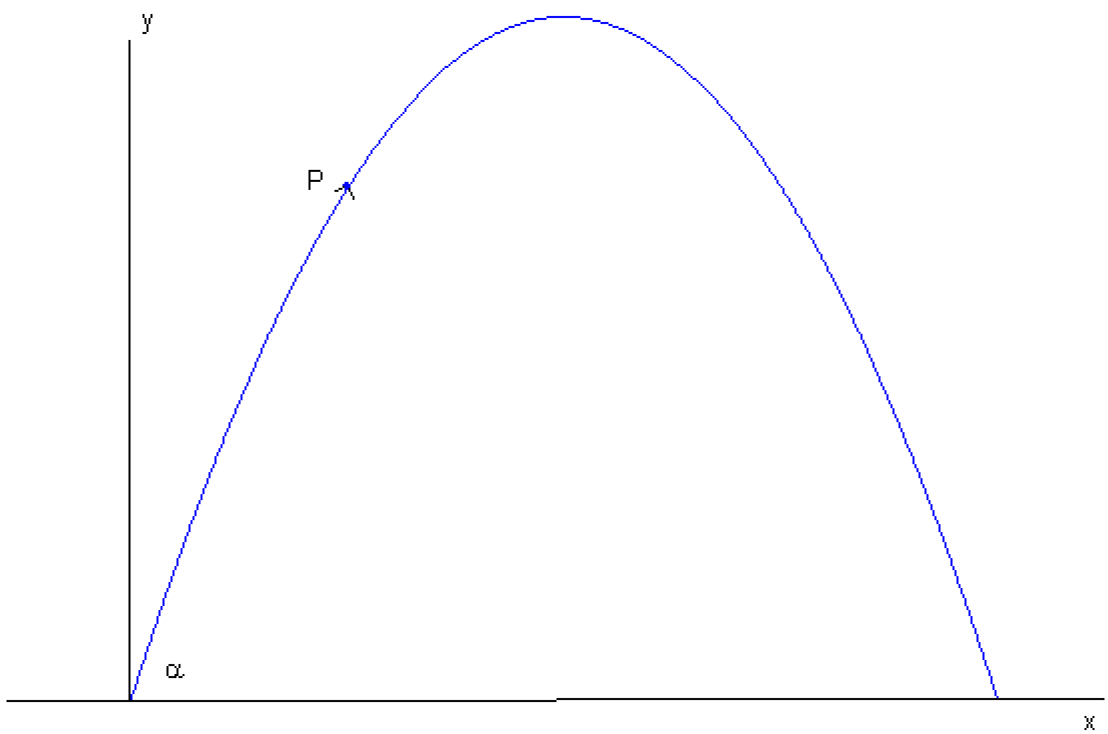
e) For every function f, $f(2x) = 2f(x)$
False: for example, let $f(x) = \sin x$. Then $f(2x) = \sin(2x) \neq 2\sin x$. In fact, $\sin(2x) = 2\sin x \cos x$.

f) For every function f, $f(x+h) = f(x) + f(h)$
False: for example, let $f(x) = \cos x$. Then $f(x+h) = \cos(x+h)$
$\neq \cos x + \cos h$.
In fact, $cos(x+h) = \cos x \cos h - \sin x \sin h$

2. Suppose a projectile P is fired with an initial speed of 200 ft/sec making an angle α with the ground $(0 < \alpha < \frac{\pi}{2})$. We put the origin at the location where the projectile was fired and assume that the only force that acting is the downward force of gravity. From physics, the parametric equations for its motion — until it hits the ground — are:

$$\begin{cases} x = (200\cos\alpha)\,t \\ y = -16t^2 + (200\sin\alpha)\,t \end{cases}$$

The graph below shows the projectile's path.



a) By eliminating the time parameter t, find an equation in rectangular coordinates for the path ("highway") that the projectile follows. What kind of curve is it?

<u>Solution</u>: From the first equation, $t = \frac{x}{200\cos\alpha}$. If we substitute expression that in the second equation, we get

$$\begin{aligned} y &= -16\left(\frac{x}{200\cos\alpha}\right)^2 + (200\sin\alpha)\left(\frac{x}{200\cos\alpha}\right) \\ &= \frac{-16}{200^2\cos^2\alpha}x^2 + \left(\frac{\sin\alpha}{\cos\alpha}\right)x. \end{aligned}$$

y is a quadratic function of x, so the graph is a parabola.

b) Determine the time t when the projectile hits the ground and how far it travels horizontally. *Are the parametric equations or the rectangular equation better here?*

Solution: The projectile is on the ground when $y = 0$, that is, when $-16t^2 + (200\sin\alpha)\,t = t(-16t + 200\sin\alpha) = 0$. The solutions are

$$t = 0 \text{ (at the beginning) and}$$
$$t = \frac{200\sin\alpha}{16} = \frac{25}{2}\sin\alpha.$$

At those times, we get

$$x = 0 \text{ (at the beginning) and}$$
$$x = (200\cos\alpha)(\tfrac{25}{2}\sin\alpha) = 2500\sin\alpha\cos\alpha \text{ (at final impact)}$$

You could solve the problem instead by starting with the rectangular equation.
The projectile is on the ground if $y = \frac{-16x^2}{200^2\cos^2\alpha} + (\frac{\sin\alpha}{\cos\alpha})x = x(\frac{-16x}{200^2\cos^2\alpha} + (\frac{\sin\alpha}{\cos\alpha})) = 0$.
That is true if $x = 0$ or if $x = (\frac{\sin\alpha}{\cos\alpha}) \cdot (\frac{200^2\cos^2\alpha}{16}) = 2500\,\sin\alpha\cos\alpha$.
The projectile returns to the ground when $x = 2500\,\sin\alpha\cos\alpha = (200\,\cos\,\alpha)\,t$, that is, $t = \frac{2500\,\sin\alpha\cos\alpha}{200\,\cos\alpha} = \frac{25}{2}\sin\alpha$.

You can answer b) using either the rectangular or parametric equations. In that sense, neither is "better." However, you might have an opinion about which method was easier.

c) For what angle α will the projectile travel the greatest horizontal distance? (*You don't need to know any calculus!*)

Solution: From b), the x-coordinate where the projectile hits the ground is $x = 2500\sin\alpha\cos\alpha$. We want to choose α to make this x as big as possible. We do a trick with a trig identity: $x = 2500\,\sin\alpha\cos\alpha = 2500 \cdot \frac{1}{2}\sin(2\alpha)$. Therefore x is largest when $\sin(2\alpha)$ is a big as possible – that is, when $\sin(2\alpha) = 1$. Since $0 < \alpha < \frac{\pi}{2}$, this means $\alpha = \frac{\pi}{4}$ ($= 45°$).

d) Suppose $\alpha = \frac{\pi}{6}$. Then you have the exact parametric equations for its path, and also (from (a)) the exact rectangular equation for the path. The point $P = (100\sqrt{3}, 84)$ is on the path. Describe how to get a "good" approximation for the slope of the tangent line to the path at that point. Is your approximation for the slope too big or too small?

Solution: For a "good" approximation to the slope of the tangent line, we need another point Q on the path that is "close" to P. *("How close" depends on "how good" an approximation you want: we'll leave those terms vague for now.)*

The parametric equations when $\alpha = \frac{\pi}{6}$ are:

$$\begin{cases} x = 200\cos\frac{\pi}{6}\,t = 200 \cdot \frac{\sqrt{3}}{2}t = 100\sqrt{3}\,t \\ y = -16t^2 + 200\sin\frac{\pi}{6}t = -16t^2 + 200 \cdot \frac{1}{2}t = -16t^2 + 100t \end{cases}$$

Therefore the projectile is at $P = (100\sqrt{3}, 84)$ when $t = 1$. A "nearby" point Q on the path is when (say) $t = 1.1$, and then the projectile is at

$$Q = (100\sqrt{3} \cdot 1.1,\ -16(1.1)^2 + 100(1.1)) = (110\sqrt{3},\ 90.64).$$

The slope of the line segment from P to Q is

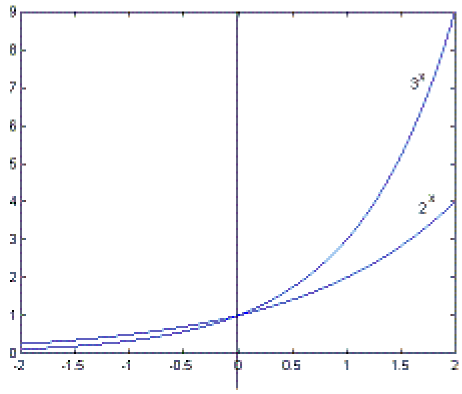
$$\frac{(90.64-84)}{110\sqrt{3}-100\sqrt{3}} = \frac{6.64}{10\sqrt{3}} \quad (\approx 0.383)$$

The curve is "arching downward" from P to Q: therefore the line segment between P and Q lies under the graph itself. This means that the tangent line at P has a slightly larger positive slope than the line segment: our approximation is too small.
(*Later in the course, we will be able to compute that the* exact *slope of the tangent line to the path at* P *is* $\frac{6.8}{10\sqrt{3}}$ (≈ 0.393)

e) Which is more useful: the parametric equations or the rectangular equation? Why?

Solution: There is, of course, no "correct answer." The rectangular equation lets you see that the path is part of a parabola. Part b) can be done using either the rectangular or parametric equations. Without using calculus, part c) seems to need the parametric form of the equations. If you wanted to know the position of projectile 1 second after firing, you'd absolutely have to have the parametric equations.

3. a) Sketch the graphs of $y = 2^x$ and $y = 3^x$.



b) What point is on the graph of every exponential function $y = a^x$?

Solution: Since $1 = a^0$, the point $(0,1)$ is always on the graph of $y = a^x$. (*See, for example, the graphs of* 2^x *and* 3^x *above.*)

c) Estimate the slope of the tangent line to $y = 2^x$ at the point $(0,1)$ using the secant line through $(0.01,\ 2^{0.01})$. Round your answer to 2 decimal places. Is the estimate larger or smaller than the actual slope of the tangent line?

Solution: The slope of the tangent line is approximated by the slope of the secant line through $(0,1)$ and $(0.01, 2^{0.01})$, that is $\frac{2^{0.01}-1}{0.01-0} \approx 0.70$. Since the secant line lies above the graph, the slope of the tangent line at $(0,1)$ is smaller than 0.70.

d) Repeat part c) for $y = 3^x$.

Solution: The slope of the tangent line is approximated by the slope of the secant line through $(0,1)$ and $(0.01, 3^{0.01})$, that is $\frac{3^{0.01}-1}{0.01-0} \approx 1.10$ Since the secant line lies above the graph, the slope of the tangent line at $(0,1)$ is smaller than 1.10.

e) For an exponential function $y = a^x$ (where $a > 0$), what happens to the slope of the tangent line at $(0,1)$ as a gets bigger?

Solution: as a increases, the slope of the tangent line at $(0,1)$ increases.

f) If the slope of the tangent line to $y = a^x$ at $(0,1)$ is 1, what do you know about the value of a ?

Solution: Based on c), d) and e), it must be that $2 < a < 3$. The particular value for a for which the tangent line to a^x at $(0,1)$ has slope exactly is denoted by the letter e. So we know that $2 < e < 3$. Since "1" is closer to 1.10 than to 0.7, it also seems likely that e is a little closer to 3 than to 2.

e is an important mathematical constant; its exact value (like the exact value of π) is an infinite non-repeating decimal. We will see later that a better approximation to the value of e is $e \approx 2.71828$.

g) Write a limit that represents the slope of the tangent line to $y = a^x$ at the point $(0, 1)$.

<u>Solution</u>: For any value of x, the corresponding point on the graph has coordinates (x, a^x). The slope of the secant line through $(0, 1)$ and (x, a^x) is $\frac{a^x - a^0}{x - 0}$. So the slope of the tangent line is given by the $\lim_{x \to 0} \frac{a^x - a^0}{x - 0} = \lim_{x \to 0} \frac{a^x - 1}{x}$.

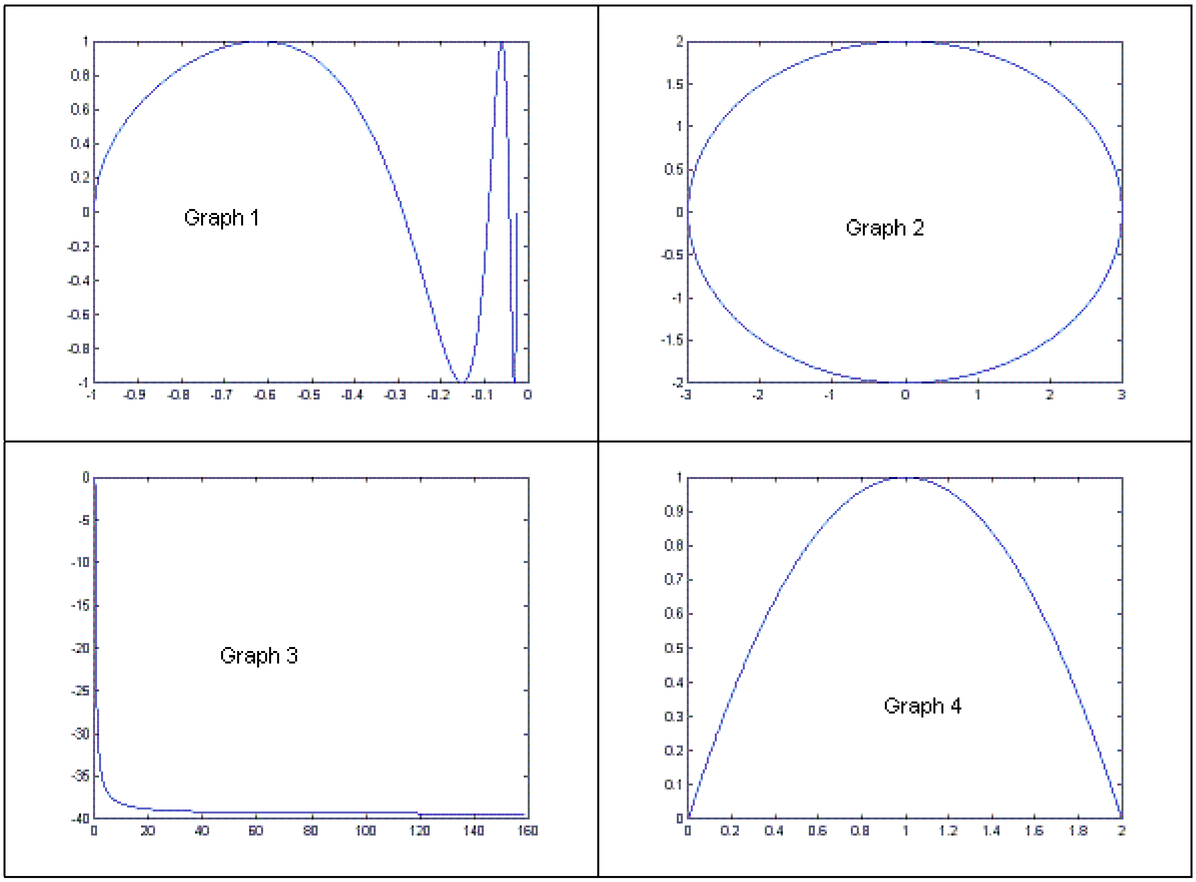
We don't know enough at this point to evaluate the limit. But

i) if $a = 2$, the value of the limit is approximately 0.70, by c)
ii) if $a = 3$, the value of the limit is approximately 1.10, by d)
iii) if $a = e$, the value of the limit is <u>exactly</u> 1, by the definition of e

4. The graphs of i) $\begin{cases} x = 3\cos t \\ y = -2\sin t \end{cases}$ ii) $\begin{cases} x = \frac{1}{2\pi - t} \\ y = -t^2 \end{cases}$

iii) $\begin{cases} x = \cos t + 1 \\ y = \sin^2 t \end{cases}$ iv) $\begin{cases} x = \frac{-1}{1+t^2} \\ y = \sin 2t \end{cases}$

are shown below in scrambled order. (*In all cases,* $0 \le t \le 2\pi$.) *Don't try to eliminate the parameter* t.



a) Match the graph with the equations and explain how you know which is which.

<u>Solution</u> (*There may be other possible correct explanations than those given below*)

For equations (i), we always have $-3 \le x \le 3$ and $-2 \le y \le 2$. That fits only Graph 2.

For equations (ii), when t gets very close to 2π, the denominator in the x-coordinate gets bigger and bigger, that is $x \to \infty$. That behavior seems to appear only in Graph 3.

For equations (iv), the x coordinate is always negative. That fits only with Graph 1.

By elimination, we match (iii) with Graph 4 (*how else could we do the match, without "elimination" ?*)

b) Eliminate the parameter in equations i) and decide in that way which graph belongs to i)

Since $\begin{cases} x = & 3\cos t \\ y = & -2\sin t \end{cases}$, we have $\begin{cases} \frac{x}{3} = \cos t \\ \frac{y}{-2} = \sin t \end{cases}$. Squaring both equations gives

$\begin{cases} \frac{x^2}{9} = \cos^2 t \\ \frac{y^2}{4} = \sin^2 t \end{cases}$. Adding, we get $\frac{x^2}{9} + \frac{y^2}{4} = \cos^2 t + \sin^2 t = 1.$

Therefore the curve is an ellipse (Graph 2).

c) Suppose in equations i) we replaced both "3" and "-2" by "4". What would the graph then be?

Solution: Using the same method, we'd get $\frac{x^2}{16} + \frac{y^2}{16} = \cos^2 t + \sin^2 t = 1$, or $x^2 + y^2 = 16$, a circle of radius 4 centered at the origin.

d) What is the graph of $\begin{cases} x = 4\cos t + 3 \\ y = 4\sin t - 2 \end{cases}$?

Solution: From c), we know that $\begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases}$
is a circle of radius 4 centered at the origin.

In $\begin{cases} x = 4\cos t + 3 \\ y = 4\sin t - 2 \end{cases}$

all x coordinates have been increased by 3, and all y coordinates have been decreased by 2. Therefore these equations represent the same circle shifted "right 3, down 2" to have center at $(3, -2)$. That circle has rectangular equation

$$(x-3)^2 + (y+2)^2 = 16.$$

(*Could you get this equation, instead, in a way similar to what we did in part b), using a trig identity?)*

5. Draw the graph of a function $f(x)$ that illustrates each of the following:

a) $\lim_{x\to 2^-} f(x)$ exists but $\lim_{x\to 2^+} f(x)$ does not exist

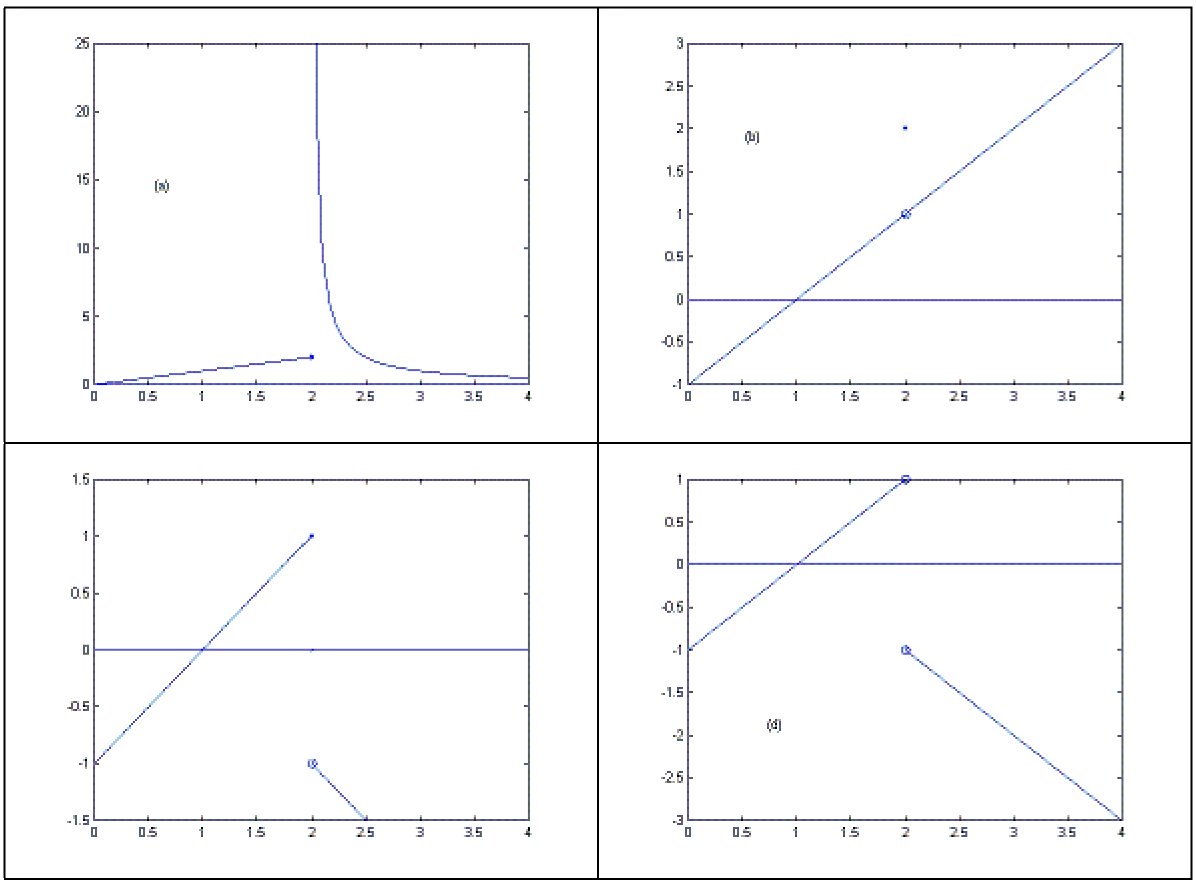
b) $\lim_{x\to 2^+} f(x) = 1 = \lim_{x\to 2^-} f(x)$ both exist but $f(2) \neq 1$

c) $\lim_{x\to 2^-} f(x) = 1,\ \lim_{x\to 2^+} f(x) = -1$, and $f(2) = 1$

d) $\lim_{x\to 2^-} f(x) = 1,\ \lim_{x\to 2^+} f(x) = -1$, and $f(2)$ is not defined

e) Does a), b), c), or d) describe the function $f(x) = \frac{|x-2|}{x-2}$? If not, then which is "most like" $f(x)$?

Solution a)-d) :



e) $f(x) = \frac{|x-2|}{x-2} = \begin{cases} \frac{x-2}{x-2} = 1 & \text{if } x > 2 \\ \frac{-(x-2)}{x-2} = -1 & \text{if } x < 2 \\ \text{undefined} & \text{if } x = 2 \end{cases}$. Therefore

$\lim_{x\to 2^+} f(x) = 1,\ \lim_{x\to 2^-} f(x) = -1$, and $f(2)$ is not defined. This is most like d): "both one-sided limits at 2 exist and have opposite values, and $f(2)$ is not defined."