Math 131, Fall 2004
Discussion Section 2 Solutions

1. Let $f(x) = \frac{(x-1)(x+2)}{(x-1)}$ and $g(x) = x + 2$

a) Are f and g the same function? Explain why or why not. Compare their graphs.

b) We know that $\lim_{x\to 1} g(x) = 3$. In light of part a), must it also be true that $\lim_{x\to 1} f(x) = 3$? Why or why not?

Solution:

a) They are not the same function since g is defined for all x but f is not defined for $x = 1$.

Canceling the $(x-1)$'s in the f formula gives the g formula, but this is only for x's where f makes sense in the first place. So, except for $x = 1$, f and g are identical. They have the same graph (a straight line) except that the f graph has a "hole" in it corresponding to $x = 1$.

$\lim_{x\to 1} f(x) = 3$ also. The only difference between the functions is at $x = 1$ – which is irrelevant in finding the limits as $x \to 1$. As $x \to 1$ $(x \neq 1)$, $f(x) = g(x)$ so both approach the limit 3.

2. Find each limit, if it exists. If it does not exist, describe the "way" in which it fails to exist.

a) $\lim_{z\to 2} \frac{z-2}{\sqrt{z+7}-3}$

Solution: $\lim_{z\to 2} \frac{z-2}{\sqrt{z+7}-3} = \lim_{z\to 2} \frac{z-2}{\sqrt{z+7}-3} \cdot \frac{\sqrt{z+7}+3}{\sqrt{z+7}+3} = \lim_{z\to 2} \frac{(z-2)(\sqrt{z+7}+3)}{(z-2)}$
$= \lim_{z\to 2} \sqrt{z+7} + 3 = 6.$

b) $\lim_{x\to 1} \frac{(x^2+2x-3)\sin(x-1)}{(x-1)^2(x+3)^2}$

Solution: $\lim_{x\to 1} \frac{(x^2+2x-3)\sin(x-1)}{(x-1)^2(x+3)^2} = \lim_{x\to 1} \frac{\sin(x-1)}{(x-1)} \cdot \frac{(x-1)(x+3)}{(x-1)(x+3)^2}$
$= \lim_{x\to 1} \frac{\sin(x-1)}{(x-1)} \cdot \frac{1}{(x+3)}.$

But $\lim_{x\to 1} \frac{\sin(x-1)}{(x-1)} = 1$ (Think of substituting $t = x - 1$ if you like. Then as $x \to 1$ we have $t \to 0$ so $\lim_{x\to 1} \frac{\sin(x-1)}{(x-1)} = \lim_{t\to 0} \frac{\sin t}{t} = 1$.) Therefore $\lim_{x\to 1} \frac{\sin(x-1)}{(x-1)} \cdot \frac{1}{(x+3)} = \frac{1}{4}$.

c) $\lim_{x\to 1}\frac{|4x-4|}{x-1}$

Solution: $\lim_{x\to 1}\frac{|4x-4|}{x-1} = 4\lim_{x\to 1}\frac{|x-1|}{x-1}$. Let $u = x-1$, so that $4\lim_{x\to 1}\frac{|x-1|}{x-1} = 4\lim_{u\to 0}\frac{|u|}{u}$.

But $\begin{cases} \lim_{u\to 0^+}\frac{|u|}{u} = \lim_{u\to 0^+}\frac{u}{u} = 1 \\ \lim_{u\to 0^-}\frac{|u|}{u} = \lim_{u\to 0^-}\frac{-u}{u} = -1. \end{cases}$ The right and left hand limits do not agree so the limit does not exist: the function has a "jump" in its graph at 1.

d) $\lim_{x\to 1}\frac{|4x-2|}{x-1}$

Solution: As $x\to 1$, $|4x-2|\to 2$ while the denominator $(x-1)\to 0$. This means that the limit does not exist. (To give more detail: $\lim_{x\to 1^+}\frac{|4x-2|}{x-1} = \lim_{x\to 1^+}\frac{4x-2}{x-1} = \infty$ and $\lim_{x\to 1^-}\frac{|4x-2|}{x-1} = \lim_{x\to 1^-}\frac{4x-2}{x-1} = -\infty$.) The function "blows up" at 1.

e) $\lim_{x\to 0}\frac{x^3\cos 3x}{x^2+x^3}$

Solution: $\lim_{x\to 0}\frac{x^3\cos 3x}{x^2+x^3} = \lim_{x\to 0}\frac{x^3\cos 3x}{x^2(1+x)} = \lim_{x\to 0}\frac{x\cos 3x}{1+x} = \frac{0\cdot 1}{1} = 0$

3. a) The correct answer to a certain question on an early Calculus I test is:

$$\lim_{x\to 3}\frac{\sin x - \sin 3}{x-3}$$

What might the question on the test have been?

Solution: The question could have been: "Write a limit that gives the slope of the tangent line to the graph of $y=\sin x$ at the point where $x=3$."

b) We don't know enough to find the exact value of this limit right now. However, explain (without using a calculator) how you know whether the value of the limit is positive or negative.

Solution: The graph of $y=\sin x$ is decreasing for $\frac{\pi}{2}\le x\le\pi$. Since $\frac{\pi}{2}\le 3\le\pi$, the limit ($=$ the slope of the tangent line at $(3,\sin 3)$) is negative.

c) Is $\lim_{x\to 3}\frac{\sin x - \sin 3}{x-3}$ smaller, larger, or equal to $\lim_{h\to 0}\frac{\sin(3+h)-\sin 3}{h}$?

Why?

<u>Solution</u>: They are equal. Suppose we substitute $x = 3 + h$ into $\lim_{h \to 0} \frac{\sin(3+h) - \sin 3}{h}$.
As $h \to 0$, we have $x \to 3$, so we get $\lim_{h \to 0} \frac{\sin(3+h) - \sin 3}{h} = \lim_{x \to 3} \frac{\sin x - \sin 3}{x - 3}$.
This means, by the way, that $\lim_{h \to 0} \frac{\sin(3+h) - \sin 3}{h}$ *also is a way to find the slope of the tangent line to* $y = \sin x$ *at the point where* $x = 3$.

4. a) Let

$$f(x) = \begin{cases} ax^3 - x^2 + a & \text{for } x < 2 \\ \frac{ax}{a-1} & \text{for } x > 2 \end{cases}$$

Find one or more values of a, if possible, which make $\lim_{x \to 2} f(x)$ exist.

b) f is not defined at 2. Is it possible to define $f(2)$ so that f becomes continuous at 2?

<u>Solutions</u>:

a) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} ax^3 - x^2 + a = 8a - 4 + a = 9a - 4$
$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{ax}{a-1} = \frac{2a}{a-1}$.

To make $\lim_{x \to 2} f(x)$ exist, we need to choose a so that

$$9a - 4 = \frac{2a}{a-1}$$

Clearing the fractions, we get the quadratic equation

$9a^2 - 13a + 4 = 2a$, so $9a^2 - 15a + 4 = 0$

The quadratic formula gives $a = \frac{15 \pm \sqrt{225 - 144}}{18} = \frac{5}{6} \pm \frac{1}{2}$, so either $a = \frac{1}{3}$ or $a = \frac{4}{3}$ will work.

b) It depends on the value of a. If $a \neq \frac{1}{3}, \frac{4}{3}$, then $\lim_{x \to 2} f(x)$ does not exist and no value you assign for $f(2)$ will make f continuous at 2.

If $a = \frac{1}{3}$, then $\lim_{x \to 2} f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = -1$, so setting $f(2) = -1$ will make f continuous at 2.

If $a = \frac{4}{3}$, $\lim_{x \to 2} f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 8$, so setting $f(2) = 8$ will make f continuous at 2.