Math 131, Fall 2004
Discussion Section 3

1. Is the line $y = 2$ a vertical asymptote for $y = f(x) = \frac{3}{x-2} - \frac{2x+8}{x^2-4}$? Are there any vertical aysmptotes? Are there horizontal asymptotes?

Solution: The line $y = 2$ is horizontal, so that line is certainly not a vertical asymptote. However, looking at the equations, the vertical lines $x = \pm 2$ might be vertical asymptotes. We want to look at the behavior of $f(x)$ as $x \to 2^+$ and $x \to 2^-$.

$f(x) = \frac{3(x+2)-(2x+8)}{(x-2)(x+2)} = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2}$.

Therefore we see that $\lim_{x \to 2} f(x) = \frac{1}{4}$ so there is no vertical asymptote at $x = 2$. However $\lim_{x \to -2^+} \frac{1}{x+2} = \infty$ and $\lim_{x \to -2^-} \frac{1}{x+2} = -\infty$, so there is a vertical asymptote at $x = -2$.

2. Let $f(x) = \frac{1-\sin x}{\cos x}$. Find the vertical asymptotes (if any) between 0 and 2π. Does the graph have any horizontal asymptotes?

Solution: If a is between 0 and 2π and $a \neq \frac{\pi}{2}$, $a \neq \frac{3\pi}{2}$, then $\lim_{x \to a} f(x) = \frac{1-\sin a}{\cos a}$. But as $x \to \frac{\pi}{2}$ and $x \to \frac{3\pi}{2}$, the denominator $\cos x \to 0$, so there might be vertical asymptotes at $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$.

As $x \to \frac{3\pi}{2}$, the numerator $1 - \sin x$ approaches 2 while the denominator approaches 0. Therefore the fraction "blows up" at $\frac{3\pi}{2}$ and we have a vertical asymptote. (Does $f(x) \to \infty$ or $-\infty$ as $x \to \frac{3\pi}{2}^-$ and $x \to \frac{3\pi}{2}^+$?)

As $x \to \frac{\pi}{2}$, both numerator and denominator approach 0, so we need somehow to "rearrange" things to try to see what happens. We can write
$f(x) = \frac{1-\sin x}{\cos x} \cdot \frac{1+\sin x}{1+\sin x} = \frac{1-\sin^2 x}{\cos x\,(1+\sin x)} = \frac{\cos^2 x}{\cos x\,(1+\sin x)}$
$= \frac{\cos x}{1+\sin x}$. Then we can see that $\lim_{x \to \frac{\pi}{2}} f(x) = \frac{0}{2} = 0$, so there is no vertical asymptote at $x = \frac{\pi}{2}$.

3. Find the horizontal and vertical asymptotes (if any) for $y = \frac{\sqrt{x^4+x^2}}{x}$

Solution: The numerator cannot approach $\pm\infty$ as $x \to a$. Therefore the only candidate for a vertical asymptote is where $x = 0$.

But $\lim\limits_{x\to 0^+} \frac{\sqrt{x^4+x^2}}{x} = \lim\limits_{x\to 0^+} \frac{x\sqrt{x^2+1}}{x} = \lim\limits_{x\to 0^+} \sqrt{x^2+1} = 1$ and
$\lim\limits_{x\to 0^-} \frac{\sqrt{x^4+x^2}}{x} = \lim\limits_{x\to 0^-} \frac{-x\sqrt{x^2+1}}{x} = \lim\limits_{x\to 0^-} -\sqrt{x^2+1} = -1.$

The function doesn't got to either ∞ or $-\infty$ at 0 from left or right. There is no vertical asymptote.

For horizontal asymptotes, we check:

$$\lim_{x\to\infty} \frac{\sqrt{x^4+x^2}}{x} = \lim_{x\to\infty} \frac{x\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \sqrt{x^2+1} = \infty \text{ (limit d.n.e.) and}$$
$$\lim_{x\to-\infty} \frac{\sqrt{x^4+x^2}}{x} = \lim_{x\to-\infty} \frac{-x\sqrt{x^2+1}}{x} = \lim_{x\to-\infty} -\sqrt{x^2+1} = -\infty \text{ (limit d.n.e.)}$$

There are no horizontal aysmptotes.

4 a) Show that the equation $\sin x = \frac{1}{2}x$ $(x > 0)$ has a root.

Solution: We need to show that there is an x for which $f(x) = \sin x - \frac{1}{2}x = 0$.
The function f is continuous and, moreover, $f(\frac{\pi}{2}) = 1 - \frac{\pi}{4} > 0$ and
$f(\pi) = 0 - \frac{\pi}{2} < 0$. Since “0” is a number between $f(\frac{\pi}{2})$ and $f(\pi)$, the Intermediate Value Theorem says that there is a number c between $\frac{\pi}{2}$ and π where $f(c) = 0$.
“c” is a root for $\sin x = \frac{1}{2}x$.

b) Let $f(x) = \tan x$. Although $f(\frac{\pi}{4}) = 1$ and $f(\frac{3\pi}{4}) = -1$, there is no x in the interval $[\frac{\pi}{4}, \frac{3\pi}{4}]$ for which $f(x) = 0$. Why doesn't this contradict the Intermediate Value Theorem?

Solution: The function $\tan x$ is not continuous on the interval $[\frac{\pi}{4}, \frac{3\pi}{4}]$ so the Intermediate Value Theorem does not apply.

5. The left 2 columns in the table below give census data for US population P from 1790 to 2000.

t *(decades)*	P *(millions)*	$y = 3.929e^{0.2984\,t}$
0 (1790)	3.929	3.929
1 (1800)	5.308	5.295
2 (1810)	7.24	7.136
3 (1820)	9.638	9.617
4 (1830)	12.866	12.962
5 (1840)	17.069	17.468
6 (1850)	23.192	23.542
7 (1860)	31.443	31.278
8 (1870)	38.588	42.759
9 (1880)	50.156	57.627
10 (1890)	62.948	77.663
11 (1900)	75.995	104.67
12 (1910)	91.972	141.06
13 (1920)	105.711	190.11
14 (1930)	122.775	256.21
15 (1940)	131.669	345.29
16 (1950)	150.697	465.35
17 (1960)	179.323	627.15
18 (1970)	203.185	845.21
19 (1980)	226	1139.01
20 (1990)	248.710	1535.15
21 (2000)	281.422	2068.93

a) By averaging two estimates (one too large, one too small), make your best estimate $P'(5)$ from this data.

One estimate: $P'(5) \approx \frac{P(5)-P(4)}{5-4} = 17.069 - 12.866 = 4.203$ (too small-why?)
Another estimate: $P'(5) \approx \frac{P(6)-P(5)}{6-5} = 23.192 - 17.069 = 6.123$ (too big-why?)

Final estimate: $\frac{4.203+6.123}{2} \approx 5.163$

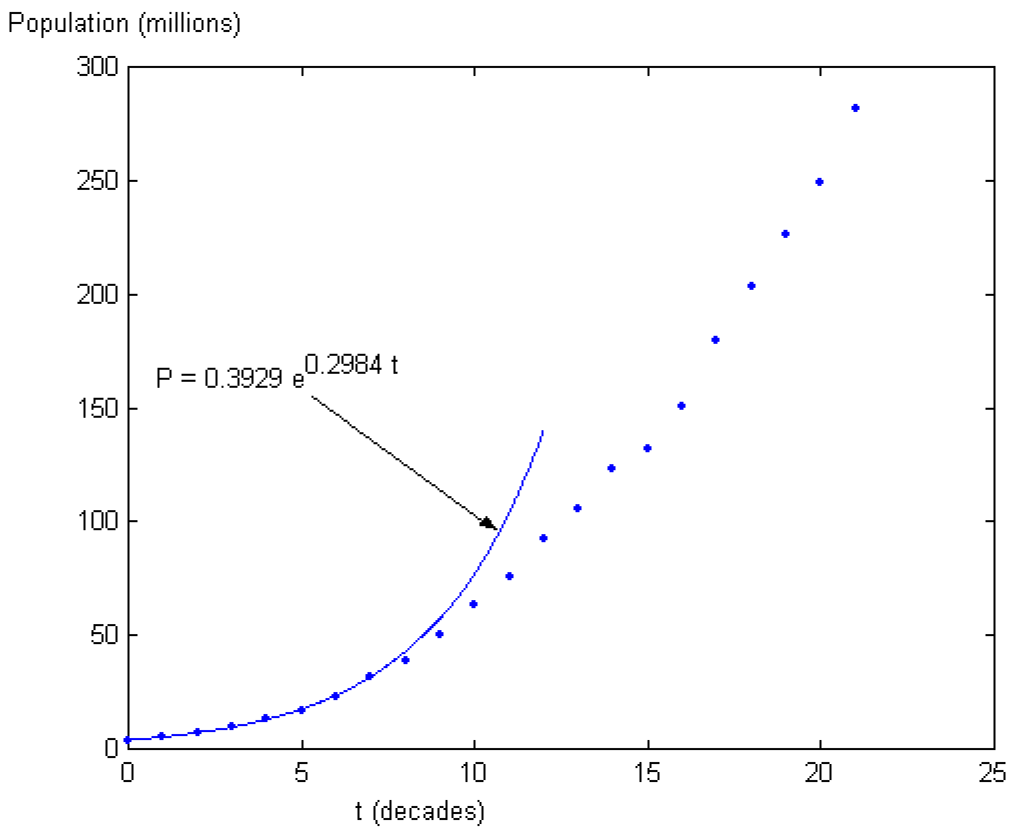
b) What does $P'(5)$ mean? What are its units?

$P'(5)$ is the rate of change of the population with respect to time at time $t = 5$ (1840). The estimated value for $P'(5)$ is 5.163 millions/decade.

c) Your estimate for $P'(5)$ can be viewed as an estimate for the value of what limit?

$$P'(5) = \lim_{h \to 0} \frac{P(5+h) - P(5)}{h}$$

d) The exponential function $y = 0.3929e^{0.2984t}$ gives a very good approximation to US population from 1790 to about 1860 – that is $P(t) \approx y = 0.3929e^{0.2984t}$ if $0 \le t \le 7$. In the figure below, what would $P'(5)$ represent?



$P'(5)$ represents the slope of the tangent line to the graph of $P(t)$ at the point where $t = 5$. We don't actually have a formula/graph for the "real" $P(t)$ but the curve pictured is a pretty good approximation for decades $0 - 7$.

e) The population growth slows down between $t = 7$ and $t = 8$. Based on the figure, which appears larger, $P'(7)$ or $P'(8)$? Do you have any ideas about why the population behaved this way?

$P'(7)$ looks larger; the <u>rate</u> of growth seems less at $t = 8$ (1870) than at $t = 7$ (1860). This was probably due to the effects of the Civil War (1861-1865). In those 4 years more than 620,000 soldiers (North and South) were killed. That's a larger number of deaths than in all the country's other wars combined up through Vietnam. As you can see from the population data, $\frac{620000}{31443000} \approx 2\%$ of the country's population in 1860. And this does not account for civilian deaths. Not only were these numbers removed from the population data, but the large number of young men killed also cut into the birth rate in the years immediately following the war.