Math 131, Fall 2004
Discussion Section 5 Solutions

1. For times $0 = \text{noon} \leq t \leq 9$ (hours), the acceleration of car moving along a straight highway is $a(t) = 30 + 15\sqrt{t}$ mi/hr^2 .

a) Let v^* be the car's velocity at 4:12 pm. Which is true: $v^* > v(4)$, $v^* = v(4)$, or $v^* > v(4)$?
Solution: 4:12 pm is time $t = 4\frac{1}{5} = 4.2$ hrs. Since $a(t) = \frac{dv}{dt} > 0$ for $0 \leq t \leq 9$, the car's velocity is always increasing. Therefore $v^* = v(4.2) > v(4)$.

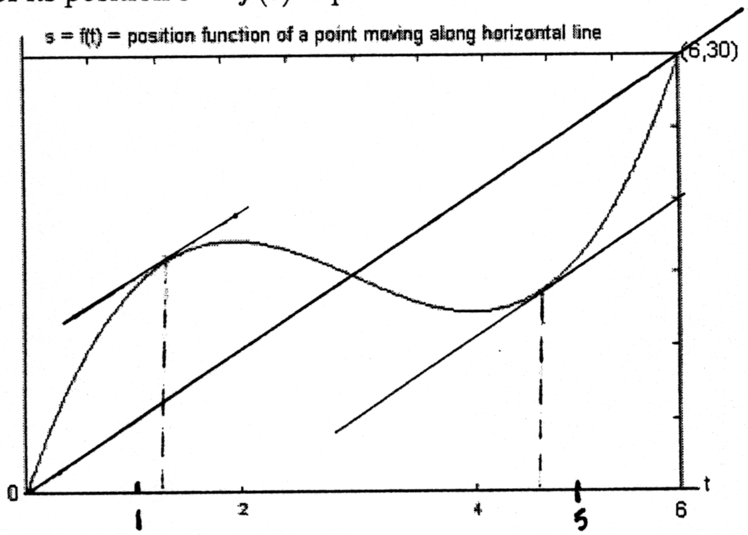
b) At time $t = 4$, the velocity of the car is 40 mi/hr. Use linear approximation to estimate v^*. Be sure your answer doesn't conflict with your answer to part a).

Solution: To make the linear approximation $L(t)$ (near time $t = 4$) we use the tangent line to
$v(t)$ at $t = 4$: $L(t) = v(4) + v'(4)(t-4) = 40 + a(4)(t-4) = 40 + 60(t-4)$.
We estimate that $v^* = v(4.2) \approx L(4.2) = 40 + 60(4.2-4) = 40 + 12 = 52$ mph.

c) Is your estimate in b) an overestimate or underestimate for v^*? How do you know this?

Solution: $\frac{d^2v}{dt^2} = \frac{da}{dt} = \frac{15}{2\sqrt{t}}$. Since $\frac{15}{2\sqrt{t}} > 0$, the graph of $v(t)$ is concave up and therefore the tangent line at $t = 4$ lies under the graph of $v(t)$. This means that $L(4.2)$ underestimates $v(4.2) = v^*$.

2. (#8, p. 179 + a bit more) A particle is moving along a horizontal straight line. The graph of its position $s = f(t)$ is pictured below:



a) When is the particle moving toward the right and when is it moving to the left?

Solution: The particle is moving in the positive direction (to the right) when the velocity $v(t) = f'(t)$ is positive, that is, when $f(t)$ is increasing. In the figure, that appears to be for $t < 2$ and for $t > 4$. The function is moving in the negative direction (to the left) when the velocity $v(t) = f'(t)$ is negative, that is, when $f(t)$ is decreasing. In the figure, that appears to be when $2 < t < 4$.

b) When does the particle have positive acceleration and when does it have negative acceleration?

Solution: The acceleration $a(t) = \frac{d^2s}{dt^2} > 0$ when $f(t)$ is concave up: this appears to be (approximately, from the picture) when $3 < t < 6$. The acceleration $a(t) = \frac{d^2s}{dt^2} < 0$ when $f(t)$ is concave down: this appears to be when $0 < t < 3$.

c) What is the average velocity of the point during the trip? Are there any times where the instantaneous velocity appears to equal the average velocity?

The average velocity during the trip is $\frac{v(6) - v(0)}{6 - 0} = \frac{30 - 0}{6 - 0} = 5$. Geometrically, this represents the slope of the line segment drawn from $(0, 0)$ to (6,30). The instantaneous velocity $= 5$ at any times t where the slope of the tangent line is 5: there appear is the figure two such places (at roughly $t = 1.25$ and $t = 4.25$)

3. For each function $y = f(x)$, write the derivative $\frac{dy}{dx}$ and the second derivative $\frac{d^2y}{dx^2}$

a) $y = f(x) = x^3 + 4x^2 - 6x + 1$

b) $f(x) = x\,(\sqrt[3]{x} + \sqrt[4]{x}\,)$

c) $f(x) = x + \frac{1}{x}$

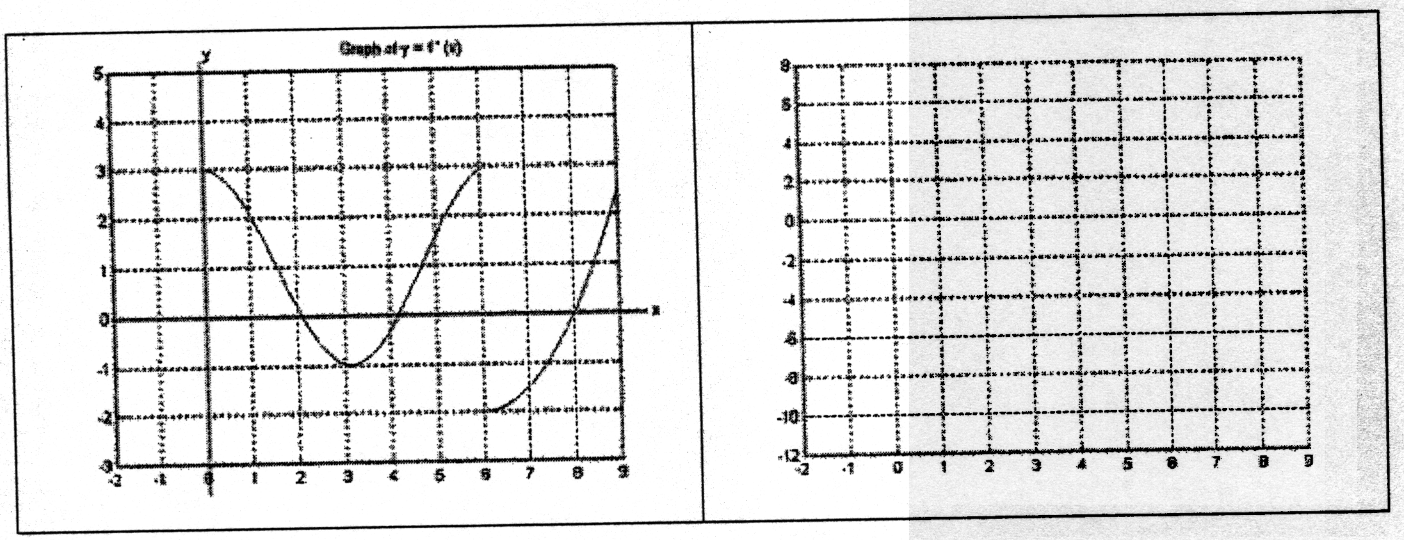
<u>Solution</u>:

a) $\frac{dy}{dx} = 3x^2 + 8x - 6, \;\; \frac{d^2y}{dx^2} = 6x + 8$

b) $y = f(x) = x(x^{1/3} + x^{1/4}) = x^{4/3} + x^{5/4}$, so
$\frac{dy}{dx} = \frac{4}{3}x^{1/3} + \frac{5}{4}x^{1/4}$ and $\frac{d^2y}{dx^2} = \frac{4}{9}x^{-2/3} + \frac{5}{16}x^{-3/4}$

c) $y = f(x) = x + x^{-1}$, so
$\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2}$ and $\frac{d^2y}{dx^2} = 2x^{-3} = \frac{2}{x^3}$.

4. (#11, p. 179) The figure shows the graph of the derivative f' of some <u>continuous</u> function f.



a) On what intervals is f increasing or decreasing?
b) At what values of x does f have a local maximum or minimum?
c) On what intervals is f concave upward or downward?
d) State the x-=coordinate(s) of the point(s) of inflection.
e) Assuming that $f(0) = 0$, sketch a graph of f on the grid provided.

Solution:

(The values given below, e.g., 2.1, are approximate readings taken from the graph.)

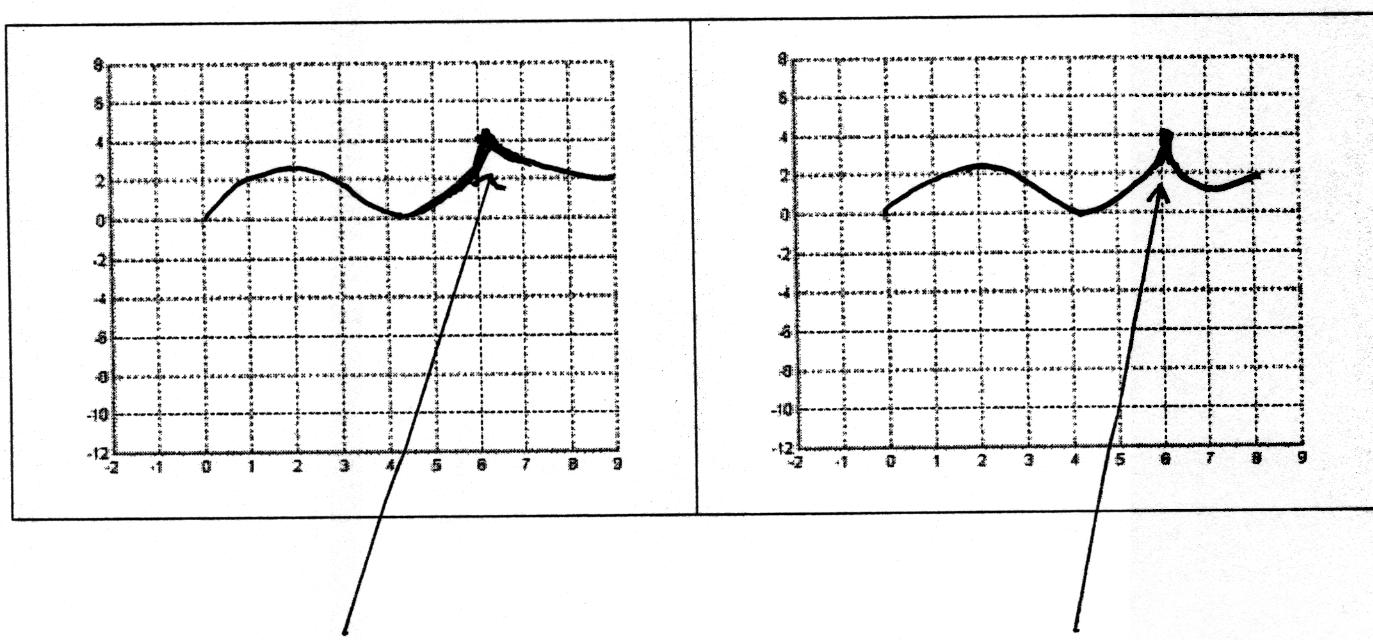
a) f is increasing where $f' > 0$, that is, for $0 < x < 2.1$, $4.2 < x < 6.1$ and $8 < x < 9$.
f is decreasing where $f' < 0$, that is, for $2.1 < x < 4.2$ (approx) and $6.1 < x < 8$.

b) f has a local min at points where f is defined and f changes from decreasing to increasing, that is, where f' switches from negative to positive: $x = 4.2$ and $x = 8$.
f has a local max at points where f is defined and f changes from increasing to decreasing, that is, where f' switches from positive to negative: $x = 2.1$ is such a point.
$f'(6)$ is not defined but f is continuous at $x = 6$ so $f(6)$ is defined. There is a local maximum there as well.

) f is concave down where $f'' < 0$, that is, where f' is decreasing: $0 < x < 3.1$
f is concave up where $f'' > 0$, that is, where f' is increasing: $3.1 < x < 6.1$ and $6.1 < x < 9$.

d) f has an inflection point where f is defined and where f changes concavity: this happens only at $x = 3.1$ (approx) where f' changes from decreasing to increasing.

e) The following show two possible graphs for f. Notice that since $f'(6)$ does not exist, there are various things that could happen to the graph of f at $x = 6$.



no derivative at 6
(corner but no vertical tangent)

No derivative at 6
(vertical tangent)