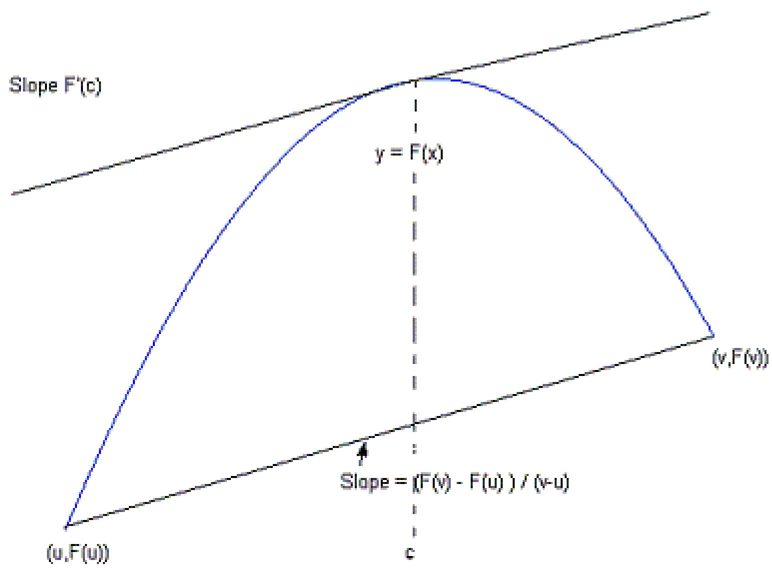
Why is the Evaluation Theorem (Fundamental Theorem of Calculus, Part II) True?

First, we want to recall the Mean Value Theorem (MVT). It states that if $F(x)$ is differentiable on an interval $[u, v]$, then there must be a point c between u and v where $\frac{F(v) - F(u)}{v - u} = F'(c)$, or, rewritten, where $F(v) - F(u) - F'(c)(v - u)$



Suppose that f is continuous on $[a, b]$ and that F is an antiderivative for f.
The "recipe" for $\int_a^b f(x)\,dx$: we subdivide $[a, b]$ into n equal parts of length Δx, choose sample points x_i^* in each subinterval (<u>how we do this is irrelevant</u>) and form the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$. Then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$. We will follow the recipe and we'll use our freedom to choose the x_i^*'s in a very clever way.

The subintervals are $[a = x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots [x_{n-2}, x_{n-1}], [x_{n-1}, x_n = b]$.

Write $F(b) - F(a) = F(x_n) - F(x_0)$

$$\begin{aligned}
&= \quad (F(x_n) - F(x_{n-1})) \\
&\quad + (F(x_{n-1}) - F(x_{n-2})) \\
&\quad + \ldots \\
&\quad + (F(x_i) - F(x_{i-1})) \\
&\quad + \ldots \\
&\quad + (F(x_2) - F(x_1)) \\
&\quad \underline{+ (F(x_1) - F(x_0))} \\
&= \sum_{i=1}^{n} F(x_i) - F(x_{i-1}).
\end{aligned}$$

Now use the MVT on each term: on $[x_{i-1}, x_i]$, $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$ for some c_i in the subinterval. But $F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x$, since $F' = f$.

So we have $F(b) - F(a) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} f(c_i)\Delta x.$

We now rename c_i as x_i^*. Then $F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*)\Delta x$, a Riemann sum where we allowed the MVT, on our behalf, to choose the sample points in each subinterval.

So, for each n, we can construct a Riemann sum in just this way. Since it doesn't matter in the definition how the x_i^*'s got chosen, we know that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

But, for each n, we constructed the Riemann sums so that for each one,

$$F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Therefore $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} F(b) - F(a) = F(b) - F(a)$ (since $F(b) - F(a)$ is a constant.

We conclude that $\int_a^b f(x)\,dx = F(b) - F(a)$, which is what we wanted to prove.