Math 131 **Exam 1**
Solutions

<u>Part</u> <u>I</u> consists of 14 multiple choice questions (worth 5 points each) and 5 true/false question (worth 1 point each), for a total of 75 points.

1. The following table gives the number of honeybees (H) in a colony at time t weeks.

t	H
0	322
1	293
2	275
3	250
4	335
5	325
6	305

What was the average rate of change of the honeybee population (honeybees/week) from $t=0$ to $t=3$? (*Round your answer to the nearest tenth.*)

A) 27.1 B) 25.3 C) 24.3 D) 19.5 E) -19.7
F) $-$ 24.0 G) -24.6 H) -21.5 I) -21.9 J) -15.4

$\frac{H(3)-H(0)}{3-0} = \frac{250-322 \text{ honeybees}}{3 \text{ weeks}} = -24.0$ **honeybees/week. The answer is negative because the population has a net decrease over this time period.**

2. (*Still using the table from Problem 1*) By averaging two estimates (one too small, the other too large), what is your best estimate from this data for the rate of change of the honeybee population (honeybees/week) when $t=2$? (*Round your answer to the nearest tenth.*)

A) 27.1 B) 25.3 C) 24.3 D) 19.5 E) -19.7
F) -24.0 G) -24.6 **H) $-$ 21.5** I) -21.9 J) -15.4

Since the population is decreasing from $t=0$ to $t=3$: $\frac{H(2)-H(1)}{2-1}$ and $\frac{H(3)-H(2)}{3-2}$ are the estimates from the nearest data to $t=2$: one too large, the other too small. Averaging them gives our "best" estimate of $\frac{1}{2}\left[\frac{H(2)-H(1)}{2-1} + \frac{H(3)-H(2)}{3-2}\right]$ $= -\frac{43}{2} = -21.5$ honeybees/week. (*Which estimate is too small and which one is too large?*)

3. A point moves according to the parametric equations $\begin{cases} x = \sqrt{3+t^2} \\ y = 7+t^2. \end{cases}$ $0 \le t \le 1.$

Exactly three of the following five statements are true. Which three are true?

i) the point is moving along part of a parabola

ii) the point moves "left to right" along its path

iii) the point is moving along part of a circle

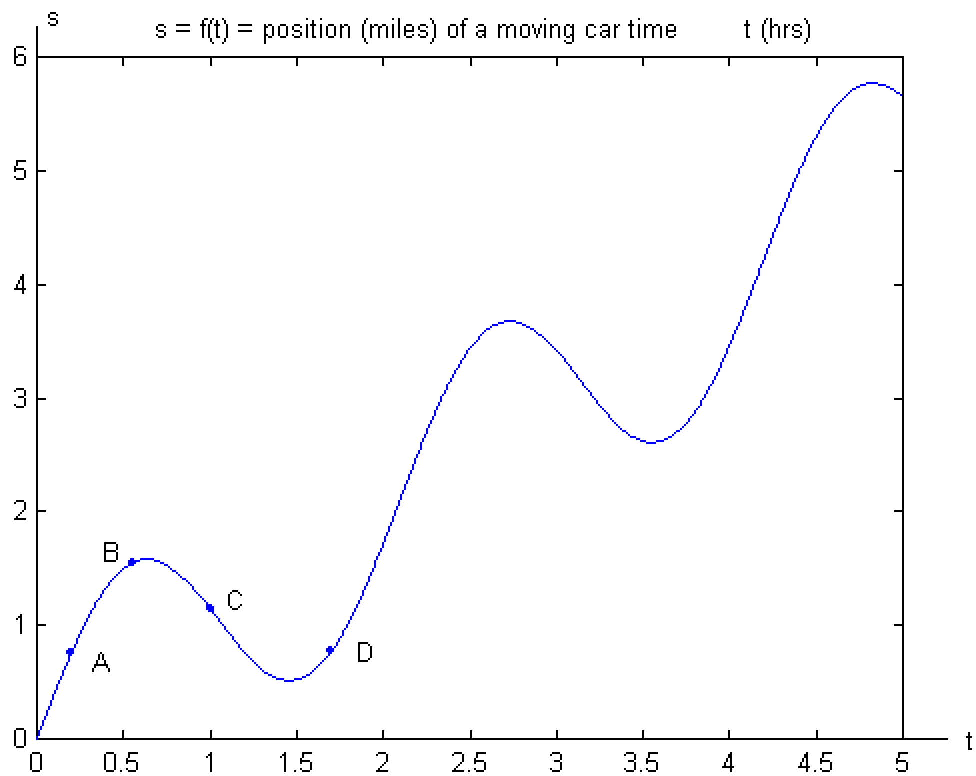
iv) the equations $\begin{cases} x = \sqrt{3+(2t)^2} \\ y = 7+(2t)^2 \end{cases}$ $0 \le t \le 1$
describe exactly the same path

v) the equations $\begin{cases} x = \frac{1}{2}\sqrt{12+u^2} \\ y = 7+\frac{u^2}{4}. \end{cases}$ $0 \le u \le 2$
describe exactly the same path.

A) iii, iv, v
B) ii, iv, v
C) ii, iii, v
D) ii, iii, iv
E) i, iv, v
F) i, iii, v
G) i, iii, iv
H) i, ii, v
I) i, ii, iv
J) i, ii, iii

i) T: Since $x^2 = 3+t^2$, we have $y = 7+(x^2-3) = 4+x^2$, which is a parabola
ii) T: As t increases, so does x — so the point is moving "rightward"
iii) F: (See i))
iv) F: these equations describe a curve ending at $(\sqrt{7}, 11)$; the given equations describe a curve ending at $(2, 8)$.
v) T: these equations are just the result of substituting $t = \frac{u}{2}$ in the given equations and adjusting the range of the parameter.

4. The following graph shows the position $s = f(t)$ of a car moving along a straight road at time t. The points A, B, C, D on the graph represent the position of the car at certain times.



Which of the following statements are <u>true</u>?

i) The car was moving faster when it was at the position corresponding to B than when it was at the position corresponding to A.

ii) The car was moving backwards for $1 \leq t \leq 1.25$

iii) The graph indicates that there were exactly 5 times (*not necessarily corresponding to A,B,C, or D*) when the car had velocity 0

iv) The car was speeding up as it moved from the position corresponding to A to the position corresponding to B.

v) During the time when the car moved from the position corresponding to A to the position corresponding to D, the average velocity was approximately 0.

A) iii, iv, v	B) ii, iv, v	**C) ii, iii, v**	D) ii, iii, iv	E) i, iv, v
F) i, iii, v	G) i, iii, iv	H) i, ii, v	I) i, ii, iv	J) i, ii, iii

Recall that the slope of the tangent line at a point $=$ the velocity at that time

i) F: The tangent line has a larger slope at A, and the slope represents the car's velocity.
ii) T: For $1 \leq t \leq 1.25$, the slope of the tangent $=$ velocity is negative
iii) T: There are exactly 5 places on the graph (tops of "peaks" or bottoms of "pits") where the tangent line is horizontal (that is, has slope 0).
iv) F: It's slowing down because the slope of the tangent line (=velocity) is decreasing between A and B
v) T: the average velocity is the slope of the secant line through A and D and that line is approximately horizontal.

5. Let $f(x) = \begin{cases} bx+5 & x<1 \\ x^2+bx+2b & x \geq 1 \end{cases}$. What value of b makes f continuous?

A) $b=1$ **B) $b=2$** C) $b=3$ D) $b=4$ E) $b=5$

F) $b=-4$ G) $b=-3$ H) $b=-2$ I) $b=-1$ J) $b=0$

For $\lim_{x\to 1} f(x)$ to exist, we need $\lim_{x\to 1^-} f(x) = b+5 = 3b+1 = \lim_{x\to 1^+} f(x)$, so $b=2$.
If $b=2$, then $\lim_{x\to 1} f(x) = 7 = f(1)$, so f is then continuous at 1.

6. Suppose the position of a point (in cm) moving along a straight line is $s = f(t) = 40 - \frac{1}{t}$ at time t (sec). What is its instantaneous velocity when $t=1$?

A) 1 cm/sec B) 2 cm/sec C) 3 cm/sec D) $\frac{2}{3}$ cm/sec

E) $\frac{1}{2}$cm/sec F) -1 cm/sec G) -2 cm/sec H) -3 cm/sec

I) $-\frac{2}{3}$ cm/sec J) $-\frac{1}{2}$ cm/sec

$$v(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(40-\frac{1}{1+h})-(40-\frac{1}{1})}{h} = \lim_{h\to 0} \frac{1-\frac{1}{1+h}}{h}$$
$$= \lim_{h\to 0} \frac{(1+h)-1}{h(1+h)} = \lim_{h\to 0} \frac{1}{1+h} = 1.$$

7. The Intermediate Value Theorem tells us that the equation $e^x = 10$ has a solution $x = c$ in the interval $[0, 3]$. By repeatedly bisecting intervals, we can find an interval as short as we like that contains c. What is the smallest number of bisections needed to enclose c in an interval of length ≤ 0.01 ?

A) 1	B) 2	C) 3	D) 4	E) 5
F) 6	G) 7	H) 8	**I) 9**	J) 10

After the 1st bisection we have 2 subintervals, $[0, 1.5]$ and $[1.5, 3]$, each of length $\frac{3}{2^1}$ and one of which contains c.
After the 2nd bisection, we have 4 intervals each of length $\frac{3}{2^2}$ and one of which contains c.
...
After the n^{th} bisection we have 2^n subintervals, each of length $\frac{3}{2^n}$ and one of which contains c.
We want to continue until the length of the subintervals $= \frac{3}{2^n} \leq 0.01$. This happens for the first time when $n = 9$. (*The easiest way to get "9" is just by trial/error using a calculator; however, you could also solve the inequality using logarithms.*)

8. The function $z = f(t) = \frac{4t+1}{\sqrt{at^2+4t-7}}$ has two horizontal asymptotes: $z = \pm\frac{1}{2}$.
What is a ?

A) 64	B) -3	C) 9	D) -9	E) $\sqrt{2}$
F) $-\sqrt{2}$	G) 0	H) 16	I) $-\frac{1}{16}$	J) $\frac{1}{64}$

$$\lim_{t\to\infty} \frac{4t+1}{\sqrt{at^2+4t-7}} = \lim_{t\to\infty} \frac{(4t+1)/t}{(\sqrt{at^2+4t-7}\,)/t} = \lim_{t\to\infty} \frac{4+\frac{1}{t}}{\sqrt{\frac{at^2+4t-7}{t^2}}}$$

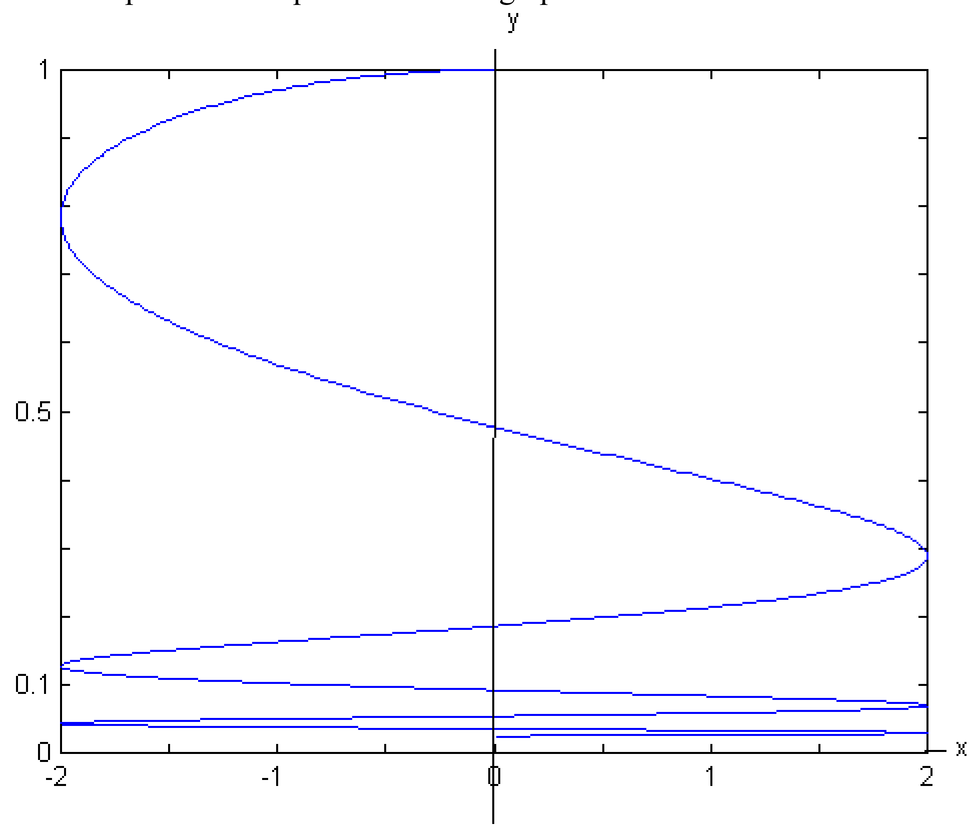
$$= \lim_{t\to\infty} \frac{4+\frac{1}{t}}{\sqrt{a+\frac{4}{t}-\frac{7}{t^2}}} = \frac{4}{\sqrt{a}} \text{ and}$$

$$\lim_{t\to-\infty} \frac{4t+1}{\sqrt{at^2+4t-7}} = \lim_{t\to-\infty} \frac{(4t+1)/t}{(\sqrt{at^2+4t-7})/t} = \lim_{t\to-\infty} -\frac{4+\frac{1}{t}}{\sqrt{\frac{at^2+4t-7}{t^2}}}$$

$$= -\lim_{t\to\infty} \frac{4+\frac{1}{t}}{\sqrt{a+\frac{4}{t}-\frac{7}{t^2}}} = \frac{-4}{\sqrt{a}}.$$

Therefore $\pm\frac{4}{\sqrt{a}} = \pm\frac{1}{2}$**, so** $a = 64$**.**

9. Which set of parametric equations has the graph shown below?



A) $\begin{cases} \boldsymbol{x = -2\sin 3t} \\ \boldsymbol{y = \frac{1}{1+t^2}} \end{cases}$ $\boldsymbol{0 \le t \le 2\pi}$

B) $\begin{cases} x = -2\sin 3t \\ y = \frac{1}{1+t^2} \end{cases}$ $0 \le t \le \pi$

C) $\begin{cases} x = \frac{1}{1+t^2} \\ y = -2\sin 3t \end{cases}$ $0 \le t \le 2\pi$

D) $\begin{cases} x = \frac{1}{1+t^2} \\ y = -2\sin 3t \end{cases}$ $0 \le t \le \pi$

E) $\begin{cases} x = 3\cos 3t \\ y = -2\sin 3t \end{cases}$ $0 \le t \le 2\pi$

F) $\begin{cases} x = 3\cos 3t \\ y = -2\sin 3t \end{cases}$ $0 \le t \le \pi$

G) $\begin{cases} x = -2\sin 3t \\ y = 3\cos 3t \end{cases}$ $0 \le t \le 2\pi$

H) $\begin{cases} x = 3\cos 3t \\ y = 3\cos 3t \end{cases}$ $0 \le t \le 2\pi$

I) $\begin{cases} x = 1+t^2 \\ y = 1-t^2 \end{cases}$ $0 \le t \le 2\pi$

J) $\begin{cases} x = 1-t^2 \\ y = 1+t^2 \end{cases}$ $0 \le t \le 2\pi$

In the graph, x ranges back and forth between ± 2. That eliminates all pairs of equations except A), B), G). Also in the graph, y is always ≥ 0. That eliminates G) For $0 \le t \le \pi$, $-2\sin 3t = 2$ only when $t = \frac{\pi}{2}$ (so that $3t = \frac{3\pi}{2}$). But in the picture $x = 2$ occurs three times. That eliminates B), leaving only A)

10. For what value of a is $\lim\limits_{x \to a} \frac{(x+3)(x^2+4x+4)}{x-a} = 1$?

A) $a = 0$ B) $a = 1$ C) $a = 2$ D) $a = 3$ E) $a = 4$
F) $a = -5$ G) $a = -4$ **H) $a = -3$** I) $a = -2$ J) $a = -1$

$\lim\limits_{x \to a} \frac{(x+3)(x^2+4x+4)}{x-a} = \lim\limits_{x \to a} \frac{(x+3)(x+2)^2}{x-a}$. Since the denominator $\to 0$ as $x \to a$, the limit can possibly exist only if the numerator $\to 0$ also as $x \to a$. That means we need either $a = -3$ or $a = -2$.

If $a = -2$, we get $\lim\limits_{x \to -2} \frac{(x+3)(x+2)^2}{x+2} = 0$.

If $a = -3$, we get $\lim\limits_{x \to -3} \frac{(x+3)(x+2)^2}{x+3} = 1$ which is what we want.

11. The size of a population of rabbits on a small island at time t years is $P(t) = \frac{35{,}000e^t}{100e^t + 900}$. As time passes, the population eventually "levels off" toward a size called the "carrying capacity" of the island (for rabbits). What is the carrying capacity for this island?

A) 100 B) 250 **C) 350** D) 900 E) 1000
F)) 2500 G) 3500 H) 9000 I) 35000 J) 100000

$\lim\limits_{t \to \infty} \frac{35{,}000e^t}{100e^t + 900} = \lim\limits_{t \to \infty} \frac{(35{,}000e^t)/e^t}{(100e^t + 900)/e^t} = \lim\limits_{t \to \infty} \frac{35000}{100 + \frac{900}{e^t}} = 350$

12. The function $f(x) = \frac{1}{(x-1)(x-2)} + \frac{1}{x(x-2)} + \frac{1}{x(x-1)}$ has how many vertical asymptotes?

A) 0 B) 1 **C) 2** D) 3 E) 4
(*Only 5 choices are intended in this problem.*)

The denominator $\to 0$ as $x \to 0$, $x \to 1$, $x \to 2$. There are no other points where $f(x)$ might $\to \pm\infty$. So we check what happens as $x \to 0$, $x \to 1$, $x \to 2$. We can rearrange the function as
$f(x) = \frac{1}{(x-1)(x-2)} + \frac{1}{x(x-2)} + \frac{1}{x(x-1)} = \frac{x + (x-1) + (x-2)}{x(x-1)(x-2)} = \frac{3(x-1)}{x(x-1)(x-2)}$ $= \frac{3}{x(x-2)}$. In this form, it is clear that the function "blows up" near $x = 0$ and $x = 2$, so there are exactly 2 vertical asymptotes.

13. For the function $y = f(t) = e^{2t} - e^{-2t}$, it turns out that $f'(t) = 2(e^{2t} + e^{-2t})$. What is the equation of the tangent line to the graph at the point where $t = 0$?

A) $y = t$ B) $y = t + 1$ C) $y = 2t + 1$ **D) $y = 4t$**
E) $y = 4t + 1$ F) $y = 4t + 2$ G) $y = (2e)t + 4$ H) $y = (4e)t + 1$
I) $y = 4t - 2$ J) $y = t + 2$

**$f'(0) = 2(e^0 + e^0) = 2(2) = 4$ gives the slope of the tangent line.
When $t = 0$, $y = f(0) = e^0 - e^0 = 0$. The line through $(0,0)$ with slope 4 has equation $(y - 0) = 4(t - 0)$ or $y = 4t$.**

14. Let $f(x) = \begin{cases} \frac{\sin k(x-1)}{x-1} & x < 1 \\ 2 + \frac{kx^2 + kx - 2k}{x-1} & x > 1 \end{cases}$. For what value of k will $\lim_{x \to 1} f(x)$ exist?

A) $k = 0$ **B) $k = -1$** C) $k = -2$ D) $k = -3$ E) $k = -4$
F) $k = -5$ G) $k = 4$ H) $k = 3$ I) $k = 2$ J) $k = 1$

**$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{\sin k(x-1)}{x-1} = \lim_{x \to 1^-} \frac{k \sin k(x-1)}{k(x-1)} = \lim_{u \to 0^-} \frac{k \sin u}{u}$ (where $u = x - 1$)
$= k \lim_{u \to 0^-} \frac{\sin u}{u} = k$.
$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 + \frac{kx^2 + kx - 2k}{x-1} = \lim_{x \to 1^+} 2 + \frac{k(x^2 + x - 2)}{x-1}$
$= \lim_{x \to 1^+} 2 + \frac{k(x-1)(x+2)}{x-1} = 2 + k(x+2) = 2 + 3k$.
To make the limit exist, we need to have $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$, that is,
$k = 2 + 3k$, so $k = -1$.**

Questions 15)-19) are "true/false" questions

15. The Intermediate Value Theorem guarantees that the equation $x^2 - 4x - 4 = 0$ has a root in the interval $[0, 4]$.

A) True **B) False**

**If $f(x) = x^2 - 4x - 4$, then $f(0) = -4$ and $f(4) = -4$. Therefore "0" is not a number between $f(0)$ and $f(4)$ so the Intermediate Value Theorem doesn't apply. (In fact, using the quadratic formula, the roots are $x = \frac{4 \pm \sqrt{16 + 16}}{2}$
$= 2 \pm 2\sqrt{2}$, both of which are <u>outside</u> the interval [0,4].**

16. $y = \frac{\tan x}{x}$ has a vertical asymptote at $x = 0$.

A) True **B) False**

$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot 1 = 1.$

17. $\lim_{h \to 0} \frac{\sqrt{4+h}-2}{h}$ gives the slope of the tangent line to $y = \sqrt{x}$ at the point $(4, 2)$.

A) True B) False

$f'(4) =$ **the slope of the tangent line at** $(4, 2) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$
$= \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$

18 . The "Squeeze Theorem" tells us that $\lim_{x \to \infty} x \sin(\frac{1}{x}) = \infty$.

A) True **B) False**

It's true that we can write $-1 \leq \sin \frac{1}{x} \leq 1$ **and conclude that** $-x \leq x \sin \frac{1}{x} \leq x$. **But as** $x \to \infty$, **this just says that** $x \sin \frac{1}{x}$ **is trapped between two quantities one of which** $\to -\infty$ **and the other of which** $\to \infty$. **This doesn't let us conclude anything about what happens to** $x \sin \frac{1}{x}$.
In fact, $\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{u \to 0} \frac{\sin u}{u}$ **(where** $u = \frac{1}{x}$**)** $= 1$.

19. Suppose $s = v(t)$ is velocity of a point moving along a straight line. For a "small" h, $\frac{v(3+h) - v(3)}{h}$ approximates the point's acceleration at time 3.

A) True B) False

The acceleration a **is the rate of change of the velocity with respect to time,** $v'(t)$.
At time $t = 3$, $a(3) = v'(3) = \lim_{h \to 0} \frac{v(3+h) - v(3)}{h}$.
This means that $\frac{v(3+h) - v(3)}{h}$ **gets close to** $v'(3)$ **as** $h \to 0$, **so** $\frac{v(3+h) - v(3)}{h} \approx v'(3)$ **for small values of** h.

Name ______________________________________ ID Number___________________

Please put your name and ID number above and on each following sheet of Part II, in case the sheets get separated during the grading process.

Part II: (25 points) In each problem, clearly show your solution in the space provided. "Show your solution" does not simply mean "show your scratch work" — you should cross out any scratch work that turned out to be wrong or irrelevant and, where appropriate, present a readable, orderly sequence of steps showing how you got the answer. Generally, a correct answer without supporting work may not receive full credit.

20. Find each limit (if the limit does not exist, explain why).

a) $\lim_{t\to 3} \frac{3t^2-27}{t^2-t-6}$

$$\lim_{t\to 3} \frac{3t^2-27}{t^2-t-6} = \lim_{t\to 3} \frac{3(t-3)(t+3)}{(t-3)(t+2)} = \lim_{t\to 3} \frac{3(t+3)}{(t+2)} = \frac{3(6)}{5} = \frac{18}{5}$$

b) $\lim_{x\to 1} \frac{(x^3-x)\sin(x-1)\sin 3(x-1)}{x(x-1)^3}$

$$\lim_{x\to 1} \frac{(x^3-x)\sin(x-1)\sin 3(x-1)}{x(x-1)^3} = \lim_{x\to 1} \frac{x(x-1)(x+1)\sin(x-1)\sin 3(x-1)}{x(x-1)^3}$$

$$= \lim_{x\to 1} \frac{(x+1)\sin(x-1)\sin 3(x-1)}{(x-1)(x-1)} = \lim_{x\to 1} (x+1)\frac{\sin(x-1)}{x-1}\frac{\sin 3(x-1)}{x-1}$$

But $\lim_{x\to 1} \frac{\sin 3(x-1)}{x-1} = \lim_{x\to 1} \frac{3\sin 3(x-1)}{3(x-1)} = 3\lim_{u\to 0} \frac{\sin u}{u}$ **(where** $u = x-1$**)** $= 3.$

Similarly $\lim_{x\to 1} \frac{\sin(x-1)}{x-1} = 1.$

Therefore $\lim_{x\to 1} (x+1)\frac{\sin(x-1)}{x-1}\frac{\sin 3(x-1)}{x-1} = 2(1)(3) = 6.$

c) $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

(See Text, p. 114) $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$

$= \lim_{t\to 0} \frac{t^2+9-9}{t^2(\sqrt{t^2+9}+3)} = \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{6}.$

21.

a) Let $f(x) = x^2 + x$. Write, in limit form, the definition of $f'(3)$. (*Substitute into the specific function* f *given here.* *<u>Do not</u> actually work out the value of the limit.)*

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2+(3+h)-(3^2+3)}{h}$$ **<u>OR</u>**

$$f'(3) = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3} \frac{x^2+x-12}{x-3}$$

(*Of course, on simplification, both limits end up having the same value*.)

b) Suppose $y = g(t) = \sqrt{t+3}$

i) Use the limit definition to find $g'(6)$

$$g'(6) = \lim_{h\to 0} \frac{g(6+h)-g(6)}{h} = \lim_{h\to 0} \frac{\sqrt{9+h}-\sqrt{9}}{h} = \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}$$
$$= \lim_{h\to 0} \frac{9+h-9}{h(\sqrt{9+h}+3)} = \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{6}$$

ii) What is the equation of the tangent line to $y = g(t)$ where $t = 6$?

$g(6) = 3$, so we want the tangent line through $(6, 3)$. Its slope is $g'(6) = \frac{1}{6}$. Therefore the tangent line has equation $(y-3) = \frac{1}{6}(t-6)$ or $y = \frac{1}{6}t + 2$.

iii) If y represents the value ($) of an investment after t years, what are the <u>units</u> of $g'(6)$?

If y is in $ and t is in years, then $g'(6)$ has units $/year (the rate of change of the value of the investment with respect to time) .