Math 131
Exam 2 Solutions, Fall 2004

<u>Part I</u> consists of 14 multiple choice questions (worth 5 points each) and 5 true/false question (worth 1 point each), for a total of 75 points. Mark the correct answer <u>on the answer card.</u> For Part I, only the answer on the card will be graded.

1. If $y = \frac{x^3}{64} + \sqrt[3]{x}$, what is $\frac{dy}{dx}|_{x=8}$?

A) $\frac{4}{3}$ **B)** $\boldsymbol{\frac{37}{12}}$ C) $\frac{49}{8}$ D) $\frac{69}{64}$ E) $\frac{75}{12}$
F) $\frac{3}{8}$ G) $\frac{7}{12}$ H) $\frac{35}{8}$ I) $\frac{127}{64}$ F) $\frac{14}{3}$

$y = \frac{1}{64}x^3 + x^{\frac{1}{3}}$, so $\frac{dy}{dx} = \frac{3}{64}x^2 + \frac{1}{3}x^{-\frac{2}{3}} = \frac{3}{64}x^2 + \frac{1}{3\sqrt[3]{x^2}}$. Therefore $\frac{dy}{dx}|_{x=8} = \frac{3}{64}(8^2) + \frac{1}{3\sqrt[3]{64}} = 3 + \frac{1}{12} = \frac{37}{12}$

2. The slope of the tangent line to $f(t) = \frac{a+e^t}{e^{2t}}$ where $t = 0$ is 5. What is a ?

A) 0 B) -1 C) -2 **D)** $\boldsymbol{-3}$ E) -4
F) 5 G) 4 H) 3 I) 2 J) 1

Using the quotient rule gives $f'(t) = \frac{e^{2t}(e^t) - (a+e^t)(2e^{2t})}{(e^{2t})^2}$.
Therefore $5 = f'(0) = \frac{1-(a+1)(2)}{1} = -2a - 1$, so $-2a = 6$, $a = -3$.

3. Suppose $F(x) = f(g(x))$ and that:

$$g(2) = 6$$
$$g'(2) = 3$$
$$f'(2) = -1$$
$$f(2) = 7$$
$$f'(6) = -4$$

what is $F'(2)$?

A) 0	B) 2	C) 21	D) 42	E) -24
F) 4	G) 18	H) -6	I) -3	**J) $\mathbf{-12}$**

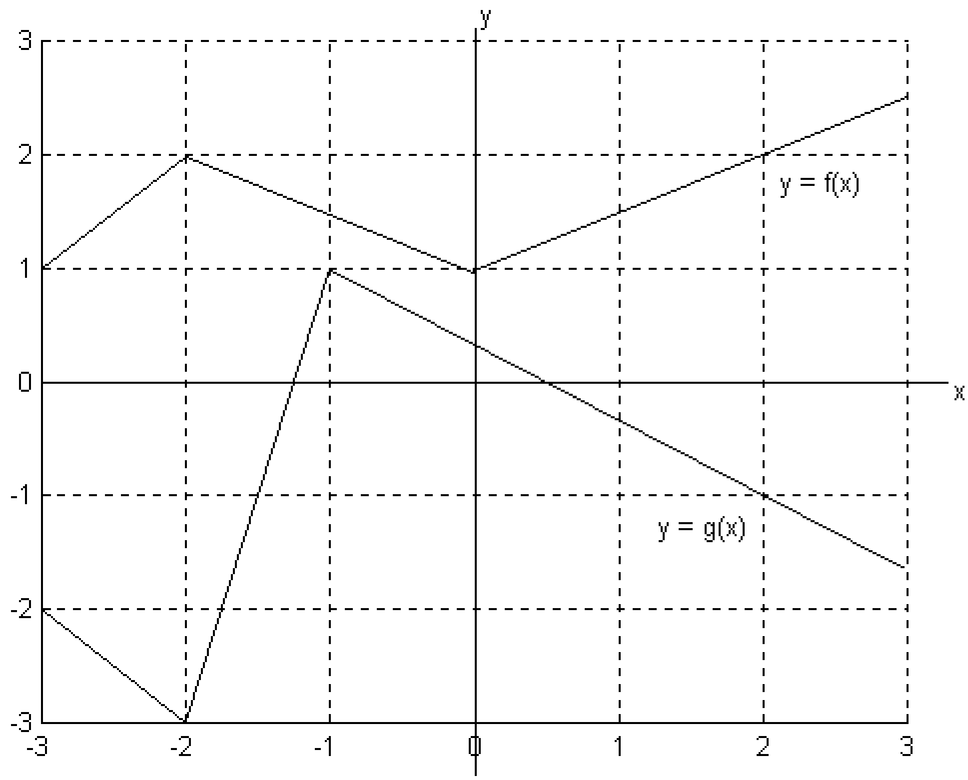
By the chain rule: $F'(x) = (f(g(x)))' = f'(g(x))g'(x)$.
So $F'(2) = (f(g(x)))'(2) = f'(g(2))g'(2)$
$= f'(6)g'(2) = (-4)(3) = -12$.

4. If $g(4) = 2$ and $g'(4) = 3$, what is $(\frac{\sqrt{x}}{g(x)})'(4)$?

A) $\frac{1}{2}$	B) $-\frac{4}{3}$	C) $\frac{14}{9}$	D) $-\frac{13}{4}$	**E) $-\frac{11}{8}$**
F) $\frac{33}{2}$	G) -4	H) $\frac{17}{6}$	I) $-\frac{17}{3}$	J) 3

Using the quotient rule gives $(\frac{\sqrt{x}}{g(x)})' = \frac{g(x)\frac{d}{dx}(\sqrt{x}) - \sqrt{x}\,g'(x)}{(g(x))^2}$
$= \frac{g(x)\cdot\frac{1}{2\sqrt{x}} - \sqrt{x}\,g'(x)}{(g(x))^2}$, so $(\frac{\sqrt{x}}{g(x)})'(4) = \frac{g(4)\cdot\frac{1}{2\sqrt{4}} - \sqrt{4}\,g'(4)}{(g(4))^2}$
$= \frac{2(\frac{1}{4}) - 2(3)}{2^2} = \frac{\frac{1}{2} - 6}{4} = \frac{-\frac{11}{2}}{4} = -\frac{11}{8}$.

5. The figure shows the graph of two functions f and g.



Let $h(x) = f(x)g(x)$. What is $h'(2)$?

A) $\frac{1}{2}$ B) $-\frac{4}{3}$ C) $\frac{14}{9}$ D) $-\frac{13}{4}$ **E)** $-\frac{11}{6}$
F) $\frac{33}{2}$ G) -4 H) $\frac{17}{6}$ I) $-\frac{17}{3}$ J) 3

From the graph, $f(2) = 2$ and $g(2) = -1$. Computing the slopes of the appropriate straight line segments, we get $f'(2) = \frac{1}{2}$ and $g'(2) = -\frac{2}{3}$. By the product rule,
$h'(2) = f(2)g'(2) + g(2)f'(2) = (2)(-\frac{2}{3}) + (-1)(\frac{1}{2}) = -\frac{4}{3} - \frac{1}{2} = -\frac{11}{6}$.

6. At time t (hrs), the size P of a certain population of bacteria is $P = 5^{t^2+t}$.
How fast is P changing at time $t = 1$? (*Round your answer to the nearest integer. All answers have the units "bacteria/hr."*)

A) 23 B) 130 C) 25 D) 18 E) 137
F) 143 **G) 121** H) 97 I) 243 J) 87

$\frac{dP}{dt} = (\ln 5)5^{t^2+t}(2t+1)$. At $t = 1$, the rate of change of P is
$\frac{dP}{dt}|_{t=1} = (\ln 5)(5^2)(3) = (\ln 5)(75) \approx 120.71 \approx 121$ (bacteria/hr).

7. A point is moving along a straight line. At time t its velocity $v(t) = t^2 - 3t + 2$. Exactly <u>two</u> of the following statements are true. Which ones are true?

i) The point is moving in the positive direction for times $t < 1$.
ii) The graph of the position function $s = f(t)$ is always concave up.
iii) During the times $8 < t < 10$, the point is speeding up.
iv) When $\frac{ds}{dt}$ is increasing, the point must be moving in the positive direction.
v) With the information given, it is possible to calculate the position at time $t = 0$

A) i, ii	**B) i, iii**	C) i, iv	D) i, v	E) ii, iii
F) ii, i	G) ii, v	H) iii, iv	I) iii, v	J) iv, v

$v = t^2 - 3t + 2 = (t-1)(t-2)$. If $t < 1$, $v > 0$, so <u>i) is true</u>.

$\frac{d^2s}{dt^2} = \frac{dv}{dt} = 2t - 3 > 0$ when $t > \frac{3}{2}$, so $f(t)$ is concave up only when $t > \frac{3}{2}$. Therefore <u>ii) is false</u>.

For times $8 < t < 10$, $v > 0$ <u>and</u> $a > 0$, so v is positive and increasing. This means the point is speeding up, so <u>iii) is true</u>.

If $v = \frac{ds}{dt}$ is increasing, that means $\frac{d^2s}{dt^2} = \frac{dv}{dt} = a > 0$, so $s = f(t)$ is concave up. But $f(t)$ could nevertheless be decreasing. Therefore iv) is <u>false</u>.

**v) is <u>false</u> because just from the velocity $v = t^2 - 3t + 2$, it is impossible to compute $s = f(t)$. For example, we can't tell whether $s = f(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t$ <u>or</u>
$s = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + 137$: both of these possible position functions have the <u>same</u> velocity function. (*If you know Tom's driving velocity at each time t, is that enough to figure out where he started? No.*)**

8. If $g(x) = \sec^2(2x)$, what is $g'(\frac{\pi}{8})$?

A) 0 B) 1 C) 2 D) $\frac{1}{2}$ E) $\sqrt{2}$
F) $\frac{1}{\sqrt{2}}$ **G) 8** H) $\frac{1}{8}$ I) $\frac{\sqrt{3}}{2}$ J) $\frac{2}{\sqrt{3}}$

$g(x) = \sec^2(2x) = (\sec(2x))^2$. Using the chain rule we get
$g'(x) = 2\,(\sec(2x))^1 \cdot \frac{d}{dx}(\sec(2x)) = 2\sec(2x) \cdot \sec(2x) \cdot \tan(2x) \cdot \frac{d}{dx}(2x)$
$= 2\sec(2x) \cdot \sec(2x) \cdot \tan(2x) \cdot (2) = 4\sec^2(2x) \cdot \tan(2x)$. So
$g'(\frac{\pi}{8}) = 4\sec^2(\frac{\pi}{4})\tan(\frac{\pi}{4}) = 4(\sqrt{2})^2(1) = 8$.

9. Suppose $f(3) = 1$ and $f'(3) = 2$. What is the estimated value, using linear approximation, for $f(2.99)$?

A) 0.93 B) 0.94 C) 0.95 D) 0.96 E) 0.97
F) 0.98 G) 0.99 H) 3 I) 1.02 J) 1.03

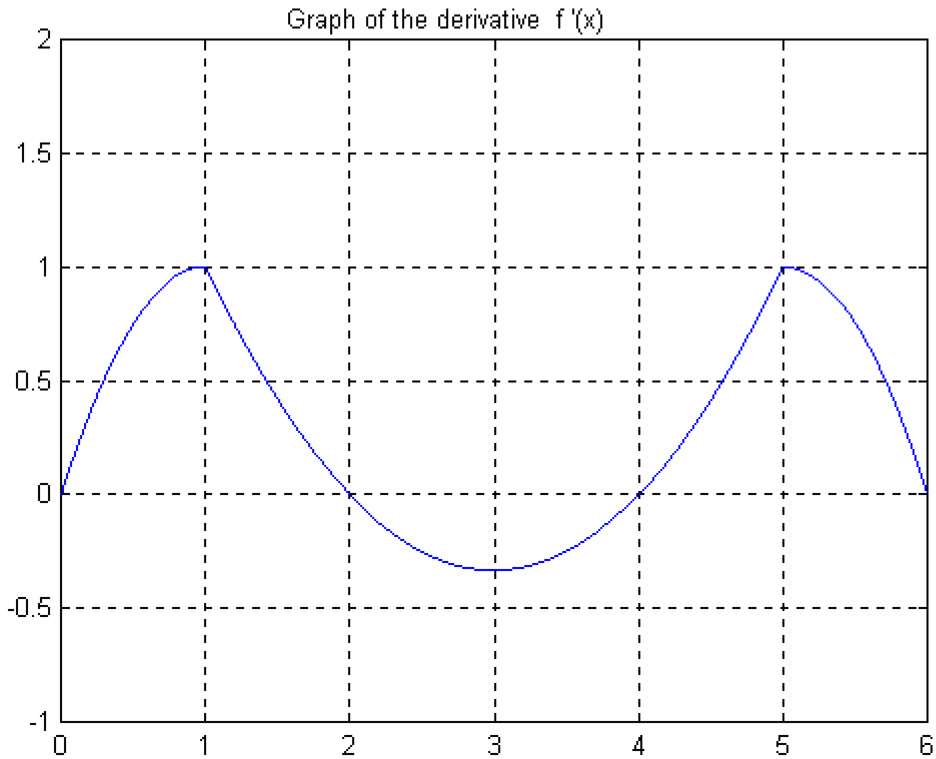
The linear approximation at 3 is an approximation for the values of $f(x)$ near 3. It is
$L(x) = f(3) + f'(3)(x-3) = 1 + 2(x-3)$. Therefore
$f(2.99) \approx L(2.99) = 1 + 2(2.99 - 3) = 1 + 2(-0.01) = 0.98$

10. The cost ($) of producing x toasters at the G.W. Crumbly Factory is $C(x) = 1\,000\,000 - 0.0001(x - 90\,000)^2$. What is the marginal cost when $x = 30\,000$? (*Round your answer to the nearest cent.*)

A) 12.87 B) 12.97 C) 14.32 D) 13.00 E) 15.67
F) 13.69 G) 18.67 H) 11.43 **I) 12.00** J) 10.45

The marginal cost is $C'(30000)$. Since $C'(x) = -0.0002(x - 90\,000)$, $C'(30\,000) = -0.0002(-60\,000) = 12$ ($). ***Note: $C'(30\,000)$ is called the marginal cost because it approximates $C(30\,001) - C(30\,000)$ = the cost of manufacturing coaster #30 001.***

11. The figure below shows the graph of the derivative $y = f'(x)$ for some function $y = f(x)$.



Exactly two of the following statements are true. Which ones are true?

i) $f(x)$ has an inflection point at $x = 3$
ii) $f(x)$ has a local maximum at $x = 4$
iii) $f''(x)$ is increasing for $5 < x < 6$
iv) $f(x)$ is concave down for $3 < x < 5$
v) $f''(x) < 0$ for $1 < x < 3$

A) i, ii	B) i, iii	C) i, iv	**D) i, v**	E) ii, iii
F) ii, iv	G) ii, v	H) iii, iv	I) iii, v	J) iv, v

At $x = 3$, the derivative $f'(x)$ switches from decreasing to increasing so at $x = 3$ $f''(x)$ changes from negative to positive. Therefore f switches concavity at $x = 3$. So i) is true.

Just to the left of $x = 4$, $f'(x) < 0$ and just to the right of $x = 4$, $f'(x) > 0$. So just to the left of $x = 4$, $f(x)$ is decreasing and just to the right of $x = 4$, $f(x)$ is increasing. So f has a local minimum at $x = 4$. Therefore ii) is false.

If $f''(x)$ were increasing for $5 < x < 6$, then $f'(x)$ would be concave up on that interval, but it isn't. So iii) is false.

Since $f'(x)$ is increasing for $3 < x < 5$ so $f''(x) > 0$ on that interval and $f(x)$ is concave up. So iv) is false.

v) For $1 < x < 3$, $f'(x)$ is decreasing so $f''(x) < 0$. Therefore v) is true.

12. Let $y = xe^{-8x^2}$. What is the smallest value of a that makes the statement "y is decreasing for $x > a$" true?

A) 8 B) 4 C) 2 D) $\frac{1}{2}$ **E) $\frac{1}{4}$**
F) 0 G) $-\frac{1}{4}$ H) $-\frac{1}{2}$ I) -1 J) -2

$f(x)$ is decreasing where $f'(x) < 0$.
$f'(x) = e^{-8x^2}(1) + x(-16xe^{-8x^2}) = e^{-8x^2} - 16x^2e^{-8x^2}$
$= e^{-8x^2}(1 - 16x^2)$.
Since $e^{-8x^2} > 0$, $f'(x) < 0$ exactly where $1 - 16x^2 < 0$, that is when $16x^2 > 1$. Solving this inequality gives $x < -\frac{1}{4}$ or $x > \frac{1}{4}$.
The statement "y is decreasing if $x > \frac{1}{4}$" is true.

13. What is $\lim_{h\to 0} \frac{(1+h)^{10000}-1}{h}$?

A) 0 **B) 10000** C) e D) 100 E) 1
F) $\frac{1}{e}$ G) $\frac{1}{100}$ H) $\frac{1}{10000}$ I) 2^{10000} J) ∞ (limit d.n.e.)

If we let $f(x) = x^{10000}$, then $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^{10000}-1}{h}$.
But we know now that $f'(x) = 10000x^{9999}$, so $f'(1) = 10000$.

14. There is one (and only one) line through the point $(2, 1)$ that is tangent to the graph of $y = \frac{x}{x-1}$ at some point P on the graph. What is the x-coordinate of P?

A) $\frac{3}{2}$ B) 1 C) 2 D) -1 E) -2
F) $\frac{1}{4}$ G) $-\frac{2}{3}$ H) $\frac{5}{3}$ I) 3 J) $\frac{3}{4}$

$f'(x) = \frac{(x-1)(1)-x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$.

If $(a, \frac{a}{a-1})$ is a point on the graph, then the slope at this point is $\frac{-1}{(a-1)^2}$. Therefore the tangent line has equation $y - \frac{a}{a-1} = \frac{-1}{(a-1)^2}(x-a)$.

If $(2, 1)$ is going to be on the tangent line, it has to satisfy the equation of the tangent line, so

$$1 - \frac{a}{a-1} = \frac{-1}{(a-1)^2}(2-a).$$

Multiplying both sides by $(a-1)^2$ gives

$$(a-1)^2 - a(a-1) = -1(2-a), \text{ or}$$
$$a^2 - 2a + 1 - a^2 + a = -2 + a, \text{ or}$$
$$2a = 3$$
$$a = \frac{3}{2}$$

Questions 15-19 are true/false questions.

15. If $\lim_{h\to 0} f(2+h) = f(2)$, then f must have a derivative at 2.

A) True **B) False**

$\lim_{h \to 0} f(2+h) = f(2)$ simply states that f is continuous at 2. (If you substitute $x = 2+h$, then $\lim_{h \to 0} f(2+h) = \lim_{x \to 2} f(x) = f(2)$.) But a function that is continuous at 2 does not have to be differentiable at 2 – consider for example, $f(x) = |x-2|$.

16. There is one and only one point where the function $f(x) = |x^2+1| + |x-2|$ fails to have a derivative.

A) True B) False

Since $x^2+1 > 0$, $f(x) = |x^2+1| + |x-2| = x^2+1+|x-2|$. This function fails to have a derivative only at $x = 2$.

17. Suppose T, P, V (the temperature, pressure, and volume of a gas in a container) are related by $\frac{PV}{T} = k$, where k is a constant. If the temperature is held constant, then $\frac{dV}{dP} = -\frac{kT}{P^2}$.

A) True B) False

Solving $\frac{PV}{T} = k$ for V gives $V = \frac{kT}{P}$. If T is held constant, then $V = kT\,\frac{1}{P}$ $= kT\,P^{-1}$ is a function of P alone, and $\frac{dV}{dP} = kT\,(-1P^{-2}) = -\frac{kT}{P^2}$.

18. Suppose $f'(x) = (x-2)^2(x-5)$. Then $f(x)$ has either a local maximum or a local minimum at $x = 2$.

A) True **B) False**

Near $x = 2$ (on the left or the right), $(x-5) < 0$. Furthermore, near $x = 2$ (on the left or the right), $(x-2)^2 > 0$. Therefore near $x = 2$ (on the left or the right), $f'(x) < 0$. This means the graph of $f(x)$ decreases as $x \to 2^-$, flattens out for a horizontal tangent $x = 2$, and then continues to decrease just to the right of $x = 2$. There is neither a local max nor min at $x = 2$.

19. If f has a derivative at 3, then $\lim_{x \to 3} f(x) = f(3)$.

A) True B) False

If f has a derivative at 3, then f must be continuous at 3, and that's just what $\lim_{x \to 3} f(x) = f(3)$ means.

Part II: (25 points) In each problem, clearly show your solution in the space provided. "Show your solution" does not simply mean "show your scratch work" — you should cross out any scratch work that turned out to be wrong or irrelevant and, where appropriate, present a readable, orderly sequence of steps showing how you got the answer. Generally, a correct answer without supporting work may not receive full credit.

20. For each function $y = f(x)$ given below, find the derivative. *After all the differentiation is completed, no further simplifications are necessary. For example, an answer that looking like*

$$\frac{dy}{dy} = \frac{2(3)\,cos(2x) - 4x(2x+1)(3)}{(x+1)(x+3) - (2)(3)x}$$

would require no further simplification.

a) $f(x) = \frac{(3x-4)^{10}}{\cos x}$

$$\boldsymbol{f'(x) = \frac{(\cos x)\cdot 10(3x-4)^9(3) - (3x-4)^{10}(-\sin x)}{\cos^2 x}}$$ **(<u>this is a sufficient answer</u>)**

$$\boldsymbol{= \frac{(3x-4)^9(30\cos x + (\sin x)(3x-4)}{\cos^2 x}}$$

b) $f(x) = 5^{\sec(\sqrt{x}\,)}$

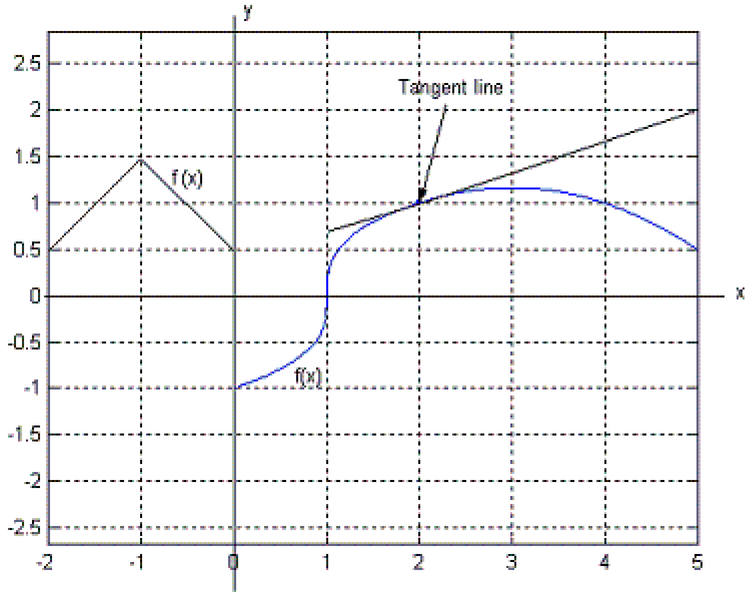
$$\boldsymbol{f'(x) = (\ln 5)\cdot 5^{\sec(\sqrt{x})}\cdot \frac{d}{dx}(\sec(\sqrt{x}))}$$
$$\boldsymbol{= (\ln 5)\cdot 5^{\sec(\sqrt{x})}\cdot \sec(\sqrt{x})\tan(\sqrt{x})\cdot \frac{d}{dx}(\sqrt{x})}$$
$$\boldsymbol{= (\ln 5)\cdot 5^{\sec(\sqrt{x})}\cdot \sec(\sqrt{x})\tan(\sqrt{x})\cdot \frac{1}{2\sqrt{x}}}$$

c) $f(x) = \tan(\sin(e^{3x^2})$

$$\boldsymbol{f'(x) = \sec^2(\sin(e^{3x^2}))\cdot \frac{d}{dx}(\sin(e^{3x^2}) = \sec^2(\sin(e^{3x^2}))\cdot \cos(e^{3x^2})\cdot \frac{d}{dx}(e^{3x^2})}$$
$$\boldsymbol{= \sec^2(\sin(e^{3x^2}))\cdot \cos(e^{3x^2})\cdot e^{3x^2}\frac{d}{dx}(3x^2)}$$
$$\boldsymbol{= \sec^2(\sin(e^{3x^2}))\cdot \cos(e^{3x^2})\cdot e^{3x^2}\cdot 6x}$$

21. The graph of a function $y = f(x)$ is shown below. On the grid beneath it, draw a reasonable graph for $f'(x)$.

Be sure your picture clearly indicates the value of $f'(2)$ (the tangent line at $x = 2$ is drawn to help you), the places where the derivative is 0, the places where the derivative *does not exist, and where $f'(x)$ is increasing or decreasing. If those things are done, then the precise shape of $f'(x)$ is not important. If you are estimating slopes, be sure to look at the scale on each axis.*



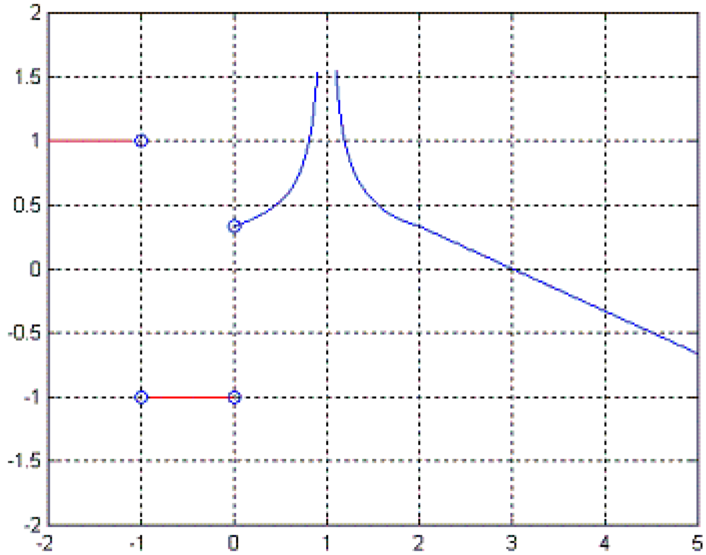
For $-2 < x < -1$, $f(x)$ has slope 1, so $f'(x) = 1$
For $-1 < x < 0$, $f(x)$ has slope -1, so $f'(x) = -1$
At $x = -1$, the graph of f has a sharp corner, so $f'(-1)$ does not exist.

$f'(0)$ does not exist because f is not continuous at 0.
Just to the right of 0, the tangent lines have a small (maybe $\approx \frac{1}{3}$) positive slope, and the slopes are positive and slopes $\to \infty$ as $x \to 1^-$ where there is a vertical tangent.
At $x = 1$, there is a vertical tangent and $f'(1)$ does not exist.

For $1 < x < 2$, the values of $f'(x)$ are positive and decreasing. At $x = 2$, we can see that the slope of the tangent line $= f'(2) = \frac{1}{3}$

For $2 < x < 3$, $f'(x)$ continues to decrease from $\frac{1}{3}$ to 0 (corresponding to a horizontal tangent at $x = 3$). Thereafter, $f'(x)$ becomes more and more negative for $3 < x < 5$.

This information is summarized in the graph below.


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1
0
1
2
3
4
5