Math 131, Exam 3 Solutions
Fall 2004

Part I consists of 14 multiple choice questions (worth 5 points each) and 5 true/false question (worth 1 point each), for a total of 75 points.

1. If $\lim\limits_{x\to 0}\frac{\cos(3x)+bx-1}{7x+\sin bx}=8$, what is b ?

A) 0 B) 1 C) 4 D) 7 E) 9
F) -2 G) -3 H) -6 **I) -8** J) -10

$\lim\limits_{x\to 0}\frac{\cos(3x)+bx-1}{7x+\sin bx}=$ "$\frac{0}{0}$", so we can use L'Hôpital's Rule to write
$\lim\limits_{x\to 0}\frac{\cos(3x)+bx-1}{7x+\sin bx}=\lim\limits_{x\to 0}\frac{-3\sin(3x)+b}{7+b\cos bx}=\frac{b}{7+b}=8$. Solving for b, we get
$b=56+8b$, so $b=-8$.

2. A spherical snowball melts in such a way its radius is decreasing at a rate of 3 cm/min at the instant when its radius is 20 cm. At that moment, how fast is its volume changing? (*Round your answer to 1 decimal place.*)

A) 12478.6 cm^3/min B) 14682.3 cm^3/min **C) -15079.6 cm^3/min**
D) -23156.7 cm^3/min E) 16783.8 cm^3/min F) -15983.7 cm^3/min
G) -14682.3 cm^3/min H) 14292.9 cm^3/min I) -12478.6 cm^3/min
J) -24978.7 cm^3/min

Volume $V=\frac{4}{3}\pi r^3$, so $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$. When $r=20$, $\frac{dr}{dt}=-3$ so
$\frac{dV}{dt}=4\pi(20)^2(-3)\approx -15079.6\,\text{cm}^3/\text{min}$.

3. A certain cone is growing in such a way that its height is always twice its radius. Use differentials to estimate how much the volume changes as the radius grows from 10 m to 10.05 m. (*Round your answer to 2 decimal places.*)

A) 31.12 m^3 B) 31.27 m^3 C) 31.57 m^3 D) 31.66 m^3 E) 31.75 m^3
F) 31.42 $\mathbf{m}^3$ G) 31.55 m^3 H) 31.59 m^3 I) 31.99 m^3 J) 31.03 m^3

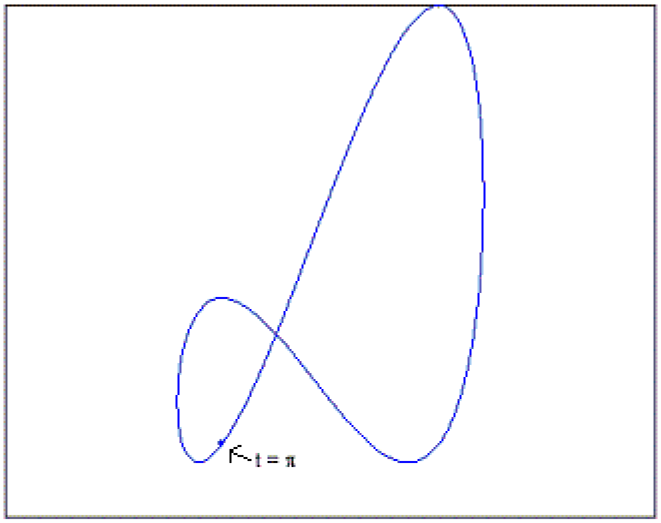
Since the height $h = 2r$, we have $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$, so $dV = 2\pi r^2 dr$. For $r = 10$ and $dr = 0.05$, we get $dV = 2\pi(10)^2(0.05)$ $\approx 31.42\,\mathrm{m}^3$

4. Suppose $f(x) = 2x^2 + x - 9$. The Mean Value Theorem states that there is a number c between 0 and 3 with a certain property. What is c?

A) 0 B) 1 C) 2 D) $\frac{5}{2}$ **E) $\frac{3}{2}$**
F) $\frac{1}{2}$ G) $\frac{5}{4}$ H) $\frac{7}{4}$ I) $\frac{9}{4}$ J) $\frac{11}{4}$

The Mean Value Theorem says that there is a point c between 0 and 3 for which $f'(c) = \frac{f(3)-f(0)}{3-0} = \frac{(18+3-9)-(-9)}{3-0} = \frac{21}{3} = 7$. Since $f'(c) = 4c + 1$, we get $4c + 1 = 7$ and therefore $c = \frac{3}{2}$.

5. What is the slope of the tangent line to the curve $\begin{cases} x = \sin t + \cos t \\ y = \sin t - \cos 2t \end{cases}$ $(0 \le t \le 2\pi)$ at the point corresponding to $t = \pi$? (*See the figure.*)



A) -1 B) $-\frac{1}{2}$ C) 0 D) $\frac{1}{4}$ E) $\frac{1}{2}$
F) $\frac{3}{4}$ **G) 1** H) $\frac{5}{4}$ I) $\frac{5}{2}$ J) 2

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t + 2\sin t}{\cos t - \sin t}$. When $t = \pi$, $\frac{dy}{dx} = \frac{-1-0}{-1+0} = 1$.

6. There are two times t in $[0, 2\pi]$ for which the curve $\begin{cases} x = \sin t + \cos t \\ y = \sin t - \cos t \end{cases}$ has a vertical tangent line. What are those times?

A) $0, \pi$ **B)** $\boldsymbol{\frac{\pi}{4}, \frac{5\pi}{4}}$ C) $\frac{\pi}{2}, \frac{3\pi}{2}$ D) $\frac{\pi}{6}, \frac{7\pi}{6}$ E) $\frac{\pi}{3}, \frac{4\pi}{3}$
F) $\frac{2\pi}{3}, \frac{5\pi}{3}$ G) $\frac{3\pi}{4}, \frac{7\pi}{4}$ H) $\frac{4\pi}{3}, \frac{7\pi}{3}$ I) $\pi, 2\pi$ J) $\frac{\pi}{3}, \frac{2\pi}{3}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t + \sin t}{\cos t - \sin t}$. We have a vertical tangent line where $\cos t - \sin t = 0$ (provided $\cos t + \sin t \neq 0$).

$\cos t - \sin t = 0$
$\cos t = \sin t$
$\tan t = 1$
$t = \frac{\pi}{4}, \frac{5\pi}{4}$

7. The point $P = (1, 1)$ is on the graph of $2x \ln y + y \ln x = 0$. What is the slope of the tangent line to the graph at P?

A) -2	B) $-\frac{3}{2}$	C) -1	**D) $-\frac{1}{2}$**	E) 0
F) $\frac{1}{2}$	G) 1	H) $\frac{3}{2}$	I) e	J) $2e$

Differentiating implicitly gives $2 \ln y + 2x \cdot \frac{1}{y}\frac{dy}{dx} + \frac{dy}{dx} \ln x + \frac{y}{x} = 0$, At $(1, 1)$ this equation becomes $2 \cdot 0 + 2 \cdot \frac{1}{1} \cdot \frac{dy}{dx} + \frac{dy}{dx} \cdot (0) + \frac{1}{1} = 0$, so $2\frac{dy}{dx} + 1 = 0$, so $\frac{dy}{dx} = -\frac{1}{2}$.

8. A rectangular box has a base in which the length is always 3 times the width. What is the largest volume for such a box if its surface area (4 sides + top + bottom) must be 1152 in^2 ?

A) 1728 in^3	B) 512 in^3	C) 695 in^3	D) 1048 in^3	E) 1246 in^3
F) 1480 in^3	G) 1624 in^3	H) 1848 in^3	I) 2142 in^3	**J) 2304 in^3**

Call the dimensions of the base x and $3x$. Let the height of the box be y. Then the volume $V = 3x^2y$. The total surface area is $1152 = 6x^2 + 8xy$, so $y = \frac{1152 - 6x^2}{8x}$.

Then $V = 3x^2(\frac{1152 - 6x^2}{8x}) = \frac{3}{8}x(1152 - 6x^2) = \frac{3}{8}(1152x - 6x^3)$, so
$V' = \frac{3}{8}(1152 - 18x^2) = 0$ gives $x^2 = \frac{1152}{18} = 64$, so (since $x > 0$) we have $x = 8$.
Since $V' > 0$ for $x < 8$ and $V' < 0$ for $x > 8$, V has an absolute maximum at $x = 8$.
For $x = 8$, we have $y = \frac{1152 - 6(64)}{64} = 12$ and $V = 3(8)^2(12) = 2304\ \text{in}^3$.

9. If $f(x) = \ln(\frac{\sqrt[4]{2x+5} \cdot (3x+5)^8}{\sqrt{6x+5}})$, what is $f'(0)$?

A) $\frac{43}{10}$	B) $\frac{21}{3}$	C) $\frac{5}{2}$	D) $\frac{41}{15}$	E) $\frac{23}{25}$
F) $\frac{14}{17}$	G) $\frac{25}{36}$	H) 0	I) $\frac{4}{3}$	J) $\frac{2}{3}$

Using properties of logarithms gives $f(x) = \frac{1}{4}\ln(2x + 5) + 8\ln(3x + 5) - \frac{1}{2}\ln(6x + 5)$,
so $f'(x) = \frac{1}{4}\frac{2}{2x+5} + 8\frac{3}{3x+5} - \frac{1}{2}\frac{6}{6x+5} = \frac{1}{2(2x+5)} + \frac{24}{3x+5} - \frac{3}{6x+5}$,
so $f'(0) = \frac{1}{10} + \frac{24}{5} - \frac{3}{5} = \frac{1+48-6}{10} = \frac{43}{10}$.

10. f is a function defined on the interval $[0,5]$, and $f(0)=f(5)=1,\ f(3)=-1$.
Suppose $f'(x)=(x-1)(x-2)^2(x-3)^3(x-4)^4$.
Then exactly <u>three</u> of the following statements are true. Which three are true?

i) f is increasing on the interval $1<x<2$
ii) f has a local min at $x=1$
iii) f has <u>neither</u> a local max nor a local min at $x=2$
iv) f has a local min at $x=3$
v) f has its absolute min at $x=3$

A) i, ii, iii B) i, ii, iv C) i, ii, v D) i, iii, iv E) i, iii, v
F) i, iv, v G) ii, iii, iv H) ii, iii, v I) ii, iv, v **J) iii, iv, v**

The following table determines the sign of $f'(x)$ on the important intervals:

	$(x-1)$	$(x-2)^2$	$(x-3)^3$	$(x-4)^4$	$f'(x)$	$f(x)$
$0<x<1$	$-$	$+$	$-$	$+$	$+$	increasing
$1<x<2$	$+$	$+$	$-$	$+$	$-$	decreasing
$2<x<3$	$+$	$+$	$-$	$+$	$-$	decreasing
$3<x<4$	$+$	$+$	$+$	$+$	$+$	increasing
$4<x<5$	$+$	$+$	$+$	$+$	$+$	increasing

$f'(x)$ is decreasing on $1<x<2$ so i) is false
$f'(x)$ is increasing on $0<x<1$ and decreasing on $1<x<2$ so f has a local maximum at $x=1$, so ii) is false
This means that iii), iv) and v) must be true
(You can also verify iii) and iv) directly, from the information in the table. The fact that $f(0)=1>f(3)=-1$ means that the local minimum at $x=3$ is also an absolute minimum (why?))

11. Suppose $f(x)=\ln((\arctan x)^3)$ for $x>0$. What is $f'(1)$? (*Note: arctan x is the "inverse tangent function" which the text sometimes also writes as tan^{-1}x.*)

A) 0 **B) $\frac{6}{\pi}$** C) $\frac{1}{\pi}$ D) $\frac{\pi}{4}$ E) $\frac{3}{\pi}$
F) $\frac{1}{2}$ G) $\frac{3}{2\pi}$ H) 2π I) $\frac{3\pi}{4}$ J) $\frac{4\pi}{3}$

$f(x)=3\ln(\arctan x)$ so $f'(x)=3\,\frac{\frac{1}{1+x^2}}{\arctan x}$, so $f'(1)=\frac{3/2}{\pi/4}=\frac{6}{\pi}$

12. On the interval $[1,3]$, the absolute minimum of the function $f(x)=\frac{x}{a}+\frac{a^2}{2x^2}$ occurs at $x=2$. What is the absolute maximum <u>value</u> of $f(x)$ on $[1,3]$?

A) $\frac{5}{2}$ B) 0 C) -2 D) $\frac{1}{2}$ E) 2
F) 3 G) $\frac{7}{2}$ H) $-\frac{3}{2}$ I) $-\frac{1}{2}$ J) 1

$f'(x) = \frac{1}{a} - \frac{a^2}{x^3}$. Since the absolute minimum occurs at $x = 2$ and since $f'(x)$ exists at every point in $(1, 3)$, it must be that $f'(2) = \frac{1}{a} - \frac{a^2}{8} = 0$, so $a = 2$ and therefore $f(x) = \frac{x}{2} + \frac{4}{2x^2} = \frac{x}{2} + \frac{2}{x^2}$. Since f has no other critical numbers in $(1, 3)$, the absolute maximum for f <u>must</u> occur at one of the two endpoints. Since $f(1) = \frac{1}{2} + \frac{2}{1} = \frac{5}{2} = 2.5$ and $f(3) = \frac{3}{2} + \frac{2}{9} = \frac{27+4}{18} = \frac{31}{18} \approx 1.72$, the absolute maximum value must be $\frac{5}{2}$.

13. For $f(x) = 3x(x-4)^{\frac{2}{3}}$, the derivative $f'(x) = (x-4)^{-\frac{1}{3}}(5x-12)$. What is the largest interval listed on which $f(x)$ is concave down? (*For your convenience: the right endpoints in the intervals listed below are increasing as you advance through the list.*)

A) $(-\infty, -\frac{5}{12})$ B) $(-\infty, 0)$ C) $(-\infty, \frac{6}{5})$ D) $(-\infty, 4^{1/3})$
E) $(-\infty, \frac{7}{4})$ F) $(-\infty, \frac{12}{5})$ G) $(-\infty, 3)$ H) $(-\infty, 4)$
I) $(-\infty, \frac{24}{5})$ J) $(-\infty, \frac{36}{5})$

$f''(x) = (x-4)^{-\frac{1}{3}}(5) + (5x-12)(-\frac{1}{3}(x-4)^{-\frac{4}{3}})$
$= (x-4)^{-\frac{4}{3}}(5(x-4) - \frac{1}{3}(5x-12))$
$= (x-4)^{-\frac{4}{3}}(5x - 20 - \frac{5}{3}x + 4) = (x-4)^{-\frac{4}{3}}(\frac{10}{3}x - 16)$
$= \frac{1}{(x-4)^{4/3}}(\frac{10}{3}x - 16)$.
$f''(x)$ does not exist for $x = 4$, and $f'(x) = 0$ for $x = \frac{24}{5}$.
We have

	$\frac{1}{(x-4)^{4/3}}$	$(\frac{10}{3}x - 16)$	$f''(x)$	$f(x)$
$x < 4$	$+$	$-$	$-$	concave down
$4 < x < \frac{24}{5}$	$+$	$-$	$-$	concave down
$\frac{24}{5} < x$	$+$	$+$	$+$	concave up

Therefore f is concave down on $(-\infty, \frac{24}{5})$ and f turns concave up to the right of $\frac{24}{5}$.

14. If $\lim_{x\to\infty}(2x+3)^{\left(\frac{1}{a\ln x}\right)} = 10$, what is a ?

A) 1 B) $\ln 2$ C) $\frac{1}{\ln 2}$ D) ln 10 E) $\ln 3$

F) $\mathbf{\frac{1}{\ln 10}}$ G) $\frac{1}{\ln 3}$ H) $\frac{1}{\ln 6}$ I) e J) $e\ln 10$

If $y = (2x+3)^{\left(\frac{1}{a\ln x}\right)}$, then $\ln y = \ln(2x+3)^{\left(\frac{1}{a\ln x}\right)} = \frac{1}{a\ln x}\ln(2x+3)$
$= \frac{\ln(2x+3)}{a\ln x}$. $\lim_{x\to\infty}\frac{\ln(2x+3)}{a\ln x}$ is of the form "$\frac{\infty}{\infty}$" so LH'ŝpitta's Rule gives
$\lim_{x\to\infty}\frac{\ln(2x+3)}{a\ln x} = \lim_{x\to\infty}\frac{\frac{2}{2x+3}}{\frac{a}{x}} = \lim_{x\to\infty}\frac{2x}{a(2x+3)} = \lim_{x\to\infty}\frac{2}{a(2+\frac{3}{x})} = \frac{1}{a}$.

Therefore $\ln y \to \frac{1}{a}$ as $x\to\infty$, and therefore $y \to e^{1/a} = 10$.
Solving for a gives $\frac{1}{a} = \ln 10$, so $a = \frac{1}{\ln 10}$.

Questions 15-19 are "true/false" questions

15. $\lim_{x\to\infty}(1+\frac{1}{x})^{6x} = \infty$

A) True **B) False**

$\lim_{x\to\infty}(1+\frac{1}{x})^{6x} = \lim_{x\to\infty}((1+\frac{1}{x})^{x})^{6} = e^{6}$

16. Mary drives the 280 miles from St. Louis to Kansas City in 5 hours. At some time during the trip she was traveling 56 miles/hr.

A) True B) False

Her average velocity during the trip was $\frac{280}{5} = 56$ mph. In this setting, the Mean Value Theorem states that there was some time c during the trip where her instantaneous velocity $=$ average velocity for the trip $= 56$ mph.

17. If c is a critical point of f and $f''(c) > 0$, then $f(x)$ has an absolute minimum at $x = c$.

A) True **B) False**

f would have a local minimum at $x = c$, but there's no reason a local minimum has to also be an absolute minimum.

18. There exists a differentiable function f such that $f(5) = 200,\ f(1) = 0$ and $f'(x) > 60$ for all x.

A) True **B) False**

Since $\frac{f(5)-f(1)}{5-1} = \frac{200}{4} = 50$, the Mean Value Theorem says there must be a number c between 1 and 5 where $f'(c) = 50$.

19. $\frac{d}{dx}\ln(8) = \frac{1}{8}$

A) True **B) False**

$\ln 8$ is a constant, so $\frac{d}{dx}\ln 8 = 0$.

Part II: (25 points) In each problem, clearly show your solution in the space provided. "Show your solution" does not simply mean "show your scratch work" – you should cross out any scratch work that turned out to be wrong or irrelevant and, where appropriate, present a readable, orderly sequence of steps showing how you got the answer. Generally, a correct answer without supporting work may not receive full credit.

20. a) Find all the critical numbers for the function $f(x) = e^x(x^2 - 3)$.

$f'(x) = e^x(x^2 - 3) + e^x(2x) = e^x(x^2 + 2x - 3)$
$= e^x(x + 3)(x - 1)$.

Since $f'(x)$ exists for all x, the only critical numbers are those x's for which $f'(x) = e^x(x + 3)(x - 1) = 0$, that is, $x = -3$ and $x = 1$.

b) What are the absolute maximum and minimum values for $f(x) = e^x(x^2 - 3)$ on the interval $[0, 2]$? (*Be sure to give the exact max and min values – although you can also include a decimal approximation for these values if you like.)*

f is continuous on the closed interval $[0, 2]$, so the Extreme Value Theorem guarantees the existence of both absolute maximum and absolute minimum values. These must occur at either an endpoint or a critical number in $(0, 2)$ – and by part a), the only such critical number is $x = 1$.

Testing at these points, we get
$$f(0) = e^0(0 - 3) = -3$$
$$f(1) = e^1(1 - 3) = -2e \approx -5.4$$
$$f(2) = e^2(4 - 3) = e^2 \approx 7.4$$
so the maximum value is e^2 and the minimum value is $-2e$.

21. a) Find $\frac{dy}{dx}$ if $y = \log_2(\frac{2x}{x^2+1})$ (*No simplification is necessary after you get to a correct formula* $\frac{dy}{dx} = ...$)

We can simplify first. $y = \log_2(\frac{2x}{x^2+1}) = \log_2 2 + \log_2 x - \log_2(x^2+1)$, so
$\frac{dy}{dx} = \frac{1}{(\ln 2)\,x} - \frac{2x}{(\ln 2)(x^2+1)}$

b) Find $\frac{dy}{dx}$ if $y = x^{\arctan x}$ (*No simplification is necessary after you get to a correct formula* $\frac{dy}{dx} = ...$)

Use logarithmic differentiation: $\ln y = \ln(x^{\arctan x}) = (\arctan x)(\ln x)$, so
$\frac{1}{y}\frac{dy}{dx} = (\arctan x)(\frac{1}{x}) + (\ln x)(\frac{1}{1+x^2}) = \frac{\arctan x}{x} + \frac{\ln x}{1+x^2}$, so
$\frac{dy}{dx} = y(\frac{\arctan x}{x} + \frac{\ln x}{1+x^2}) = x^{\arctan x}(\frac{\arctan x}{x} + \frac{\ln x}{1+x^2})$

c) Find $\lim\limits_{x\to 0^+}(\frac{1}{x} - \csc x)$

(*Note: this exact problem was done in the lecture.*)

$\lim\limits_{x\to 0^+}(\frac{1}{x} - \csc x)$ is of the form "$\infty - \infty$". We can rewrite this as the limit of a fraction:
$\lim\limits_{x\to 0^+}(\frac{1}{x} - \csc x) = \lim\limits_{x\to 0^+}(\frac{1}{x} - \frac{1}{\sin x}) = \lim\limits_{x\to 0^+}\frac{\sin x - x}{x\sin x}$. This limit is of the form "$\frac{0}{0}$". We use LH'ŝpitta's rule (twice): $\lim\limits_{x\to 0^+}\frac{\sin x - x}{x\sin x} = \lim\limits_{x\to 0^+}\frac{\cos x - 1}{x\cos x + \sin x} = \lim\limits_{x\to 0^+}\frac{\cos x - 1}{x\cos x + \sin x}$
$= \lim\limits_{x\to 0^+}\frac{-\sin x}{\cos x - x\sin x + \cos x} = \frac{0}{2} = 0$