Why is the Fundamental Theorem of Calculus, Part I, True?

The following is an informal argument about why the Fundamental Theorem of Calculus, Part I, is true. It states:

Suppose f is a continuous function and that F is defined by $F(x) = \int_a^x f(t)\,dt$ (where a is some constant). Then $F'(x) = f(x)$.

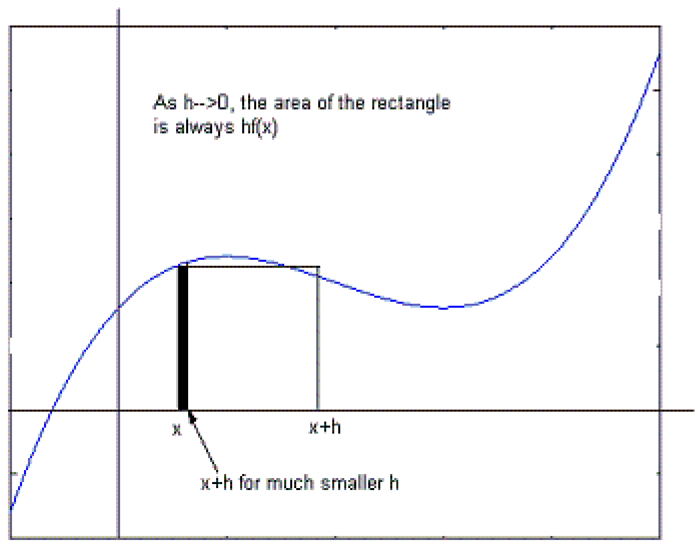
Why? To compute $F'(x)$, we go back to the definition of a derivative.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}.$$

But the numerator $\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$
(because $\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt$).

Therefore

$F'(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$. Now look at the figure below:



As $h \to 0$, $f(x+h) \to f(x)$ (because f is continuous)
and therefore
Area under graph over $[x, x+h] = \int_x^{x+h} f(t)\,dt \to$ area of rectangle $= hf(x)$.

So $F'(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h \to 0} \frac{hf(x)}{h} = f(x)$.

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Why is f required to be continuous?

If f is not continuous, the argument doesn't work: in the figure below, the region under the graph over $[x, x+h]$ doesn't come toward coinciding with the rectangle as $h \to 0$. Therefore replacing $\int_x^{x+h} f(t)\,dt$ with $hf(x)$, as we did above, isn't justified.