Review of exponential and logarithm functions

(*See text material in §1.5 and 1.6*)

Functions f and g are called <u>inverses</u> to each other if each one "undoes" the effect of the other, that is : for a in the domain of f and b in the domain of g,

1) if $f(a) = b$, then $g(b) = a$ (so that $g(f(a)) = a$)
2) if $g(b) = a$, then $f(a) = b$ (so that $f(g(b)) = b$

For example, $f(x) = 2x$ and $g(x) = \frac{1}{2}x$ are inverses:

$$g(f(a)) = g(2a) = \tfrac{1}{2}(2a) = a \quad \text{and} \quad f(g(b)) = f(\tfrac{1}{2}b) = 2(\tfrac{1}{2}b) = b.$$

For inverse functions f and g, 1) and 2) tell us that:

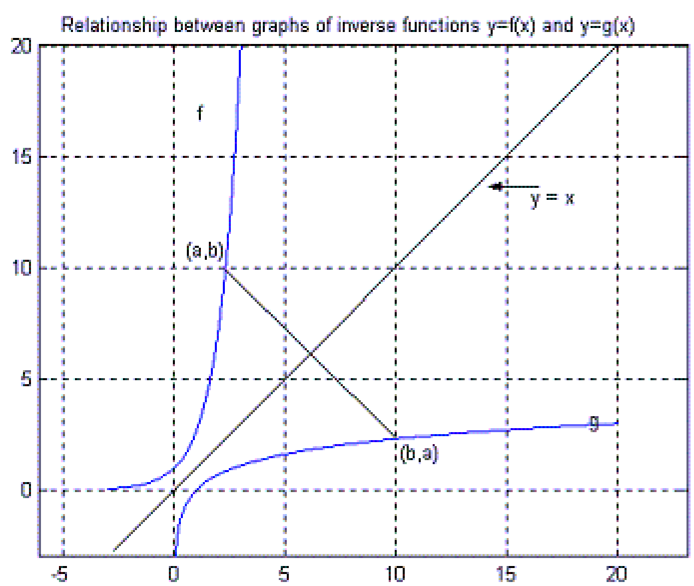
domain f = range g domain g = range f

if (a, b) is on the graph of f (because $f(a) = b$),
then (b, a) is on the graph of g (because $g(b) = a$)

and vice-versa,

if (b, a) is on the graph of g (because $g(b) = a$),
then (a, b) is on the graph of f (because $f(a) = b$)

(*See the figure*)

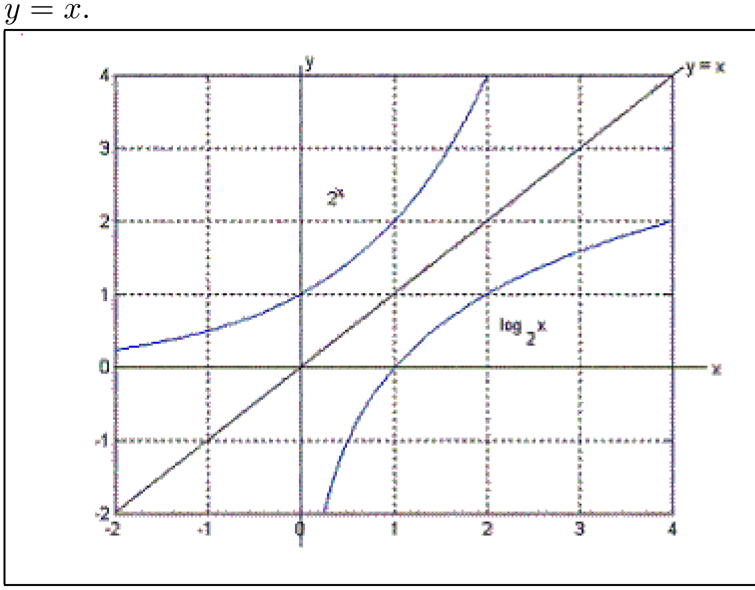


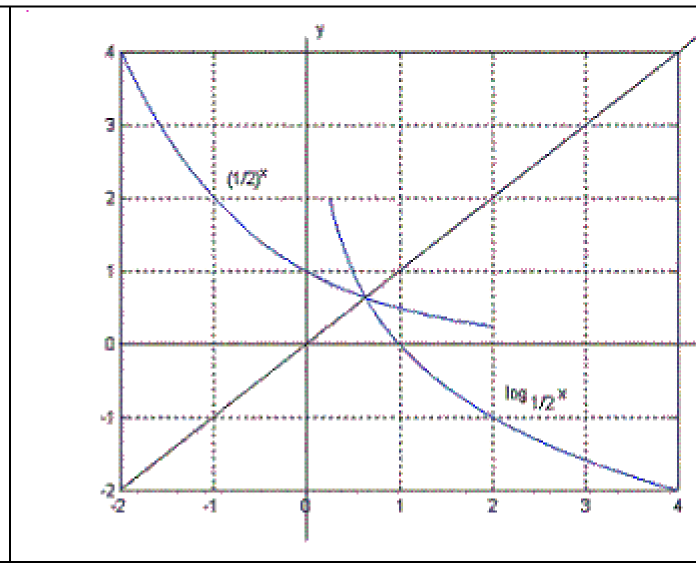
The figure illustrates how (a, b) and (b, a) are related: each is the reflection of the other across the line $y = x$. Therefore the graphs of inverse functions f and g are reflections of each other across the line $y = x$. (*Carefully draw the graphs of* $f(x) = 2x$ *and* $g(x) = \frac{1}{2}x$ *to see another example of this "reflection" property of inverse functions.*)

Exponential functions and logarithm functions are inverses of each other. The inverse of $f(x) = a^x$ (exponential, base a) is $g(x) = \log_a x$ (logarithm, base a). (*Here,* $a > 0$ *and* $a \neq 1$.

Since 1^x *is constant we don't want to call it an "exponential" function; and we want to avoid functions like* $(-2)^x$ *because they aren't defined for certain* x*'s – for example,* $(-2)^{1/2} =$ *?*

The two figures below show (on the left) the graphs of 2^x and $\log_2 x$ and (on the right) $(\frac{1}{2})^x$ and $\log_{\frac{1}{2}} x$ (*Most often we will be using base* $a > 1$*, so usually the graph of* $\log_a x$ *will be shaped similarly to that of* $\log_2 x$.). The graphs are reflections of each other across the line $y = x$.





The domain of $f(x) = a^x$ is the set of all real numbers $(-\infty < x < \infty)$ and the range is the set of all positive real numbers: $x > 0$).

Therefore the domain for the inverse function $g(x) = \log_a x$ is the set of positive real numbers and its range = set of all real numbers.

Because $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions, we have (for x's in the domain of the appropriate function) $f(g(x)) = x$ and $g(f(x)) = x$, that is

$$(*) \quad a^{\log_a x} = x \quad \text{and} \quad \log_a(a^x) = x$$

For example, $10^{\log_{10} 13} = 13$ and $\log_5(5^9) = 9$.

We can think of the equation $a^{\log_a x} = x$ as saying: " $\log_a x$ is the power (exponent) to which a must be raised to give x." If we put it that way,

> $\log_{10} 100 = 2$ because 2 is the exponent to which 10 must be raised to give 100
> $\log_3 81 = 4$ because 4 is the exponent to which 3 must be raised to give 81.
> $\log_a 1 = 0$ because $a^0 = 1$. (*Therefore* $(1, 0)$ *is on the graph of* $y = \log_a x$ *for any base* a; *we should have known this anyway – because* $(0, 1)$ *is on the graph of* $y = a^x$.

Since logarithms can be thought of as exponents, they have properties similar to rules of exponents. For example:

$\log_a(xy) = ?$

This is asking "$a^? = xy$" ? The answer is $\log_a x + \log_a y$ because the laws of exponents say $a^{\log_a x + \log_a y} = a^{\log_a x} \cdot a^{\log_a y} = xy$.
Therefore

$$\boldsymbol{\log_a xy = \log_a x + \log_a y}$$

$\log_a(x^r) = ?$

This is asking "$a^? = x^r$" ? The answer is $r \log_a x$ because the laws of exponents say $a^{r \log_a x} = (a^{\log_a x})^r = x^r$.
Therefore

$$\boldsymbol{\log_a(x^r) = r \log_a x}$$

Examples:

1) $\log_3(48) = \log_3(3 \cdot 16) = \log_3 3 + \log_3(16) = 1 + \log_3 16$
$\log_{10}(400) = \log_{10}(10^2) + \log_{10}(4) = 2 + \log_{10} 4$

2) To solve $10^{3x^2 - 4} = 7$, we can "kill" the exponential as follows:

$\log_{10}(10^{3x^2-4}) = \log_{10} 7$
$3x^2 - 4 = \log_{10} 7$
$x^2 = (4 + \log_{10} 7)/3$
$x = \pm\sqrt{(4 + \log_{10} 7)/3}$

3) To solve $\log_9(7x - 3) = 10$, we can "kill" the logarithm as follows:

$9^{\log_9(7x-3)} = 9^{10}$
$7x - 3 = 9^{10}$
$x = (9^{10} + 3)/7$

The base most often used in calculus is base $a = e$. As we know, e is a constant whose value is approximately 2.71828. The exponential function $y = e^x$ has $y = \log_e x$ as its inverse. Since we most often use base e, $\log_e x$ is given a simpler name: $\log_e x = \ln x$.

We can use properties of logarithms to see how to convert logs from one base to another.

The formula is:

$$(**) \quad \boldsymbol{\log_b x = \frac{\log_a x}{\log_a b}}$$

(In remembering this: imagine you already know how to compute logs in base a and want to compute logs in a new base b instead: the "new thing", $\log_b x$ is given in terms of the "old" things $-$ both logs on the right side of the equation use base a.

Why is (**) true?

Equation (***) is the same as saying:

$$(\log_b x)(\log_a b) = \log_a x$$

But $\log_a x$ is "the power to which a must be raised to give x.
Therefore, to prove *** is true, we need to check <u>whether</u>

$$a^{(\log_b x)(\log_a b)} = x.$$

But this <u>is</u> true because of laws of exponents:

$$a^{(\log_b x)(\log_a b)} = \left(\left(a^{\log_a b}\right)\right)^{(\log_b x)}$$
$$= b^{\log_b x} = x.$$

Examples

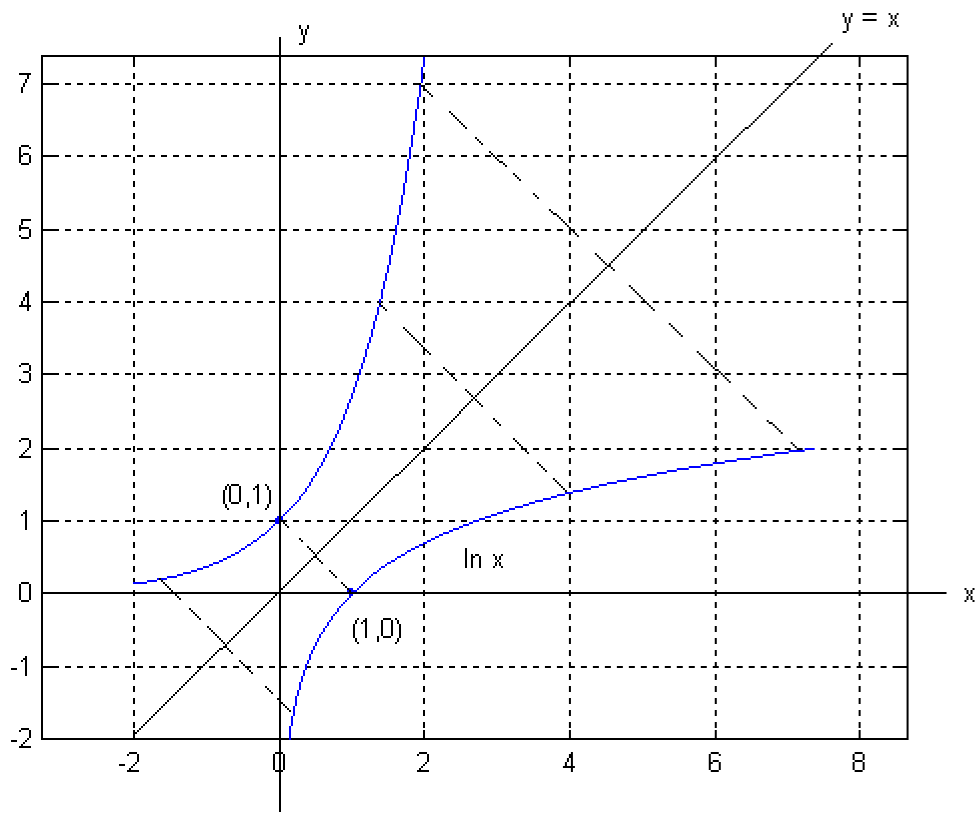
$$\log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}$$

Therefore if your calculator can do "ln", you can also use it to find the value of, say, $\log_{17} 35 = \frac{\ln 35}{\ln 17} \approx 1.2549$

For base $a = e$, the earlier equations (*) become:

$$e^{\ln x} = x \quad \text{and } \ln\left(e^{x}\right) = x$$

The graphs look like:



Notice how in reflecting across the line $y = x$, the <u>horizontal</u> asymptote (x-axis) of $y = e^x$ turns into a <u>vertical</u> asymptote (the y-axis) for the graph of the inverse $y = \ln x$.

We know that for $f(x) = e^x$, $f'(x) = f''(x) = e^x$. Since the first and second derivatives are both > 0 always, e^x is always increasing and concave up (*as the graph of* e^x *seems to show.*).

We will see soon that for $g(x) = \ln x$:

$$g'(x) = \frac{1}{x}, \qquad \text{and so}$$
$$g''(x) = -\frac{1}{x^2}$$

Since $g(x)$ is only defined for $x > 0$, $\frac{1}{x}$ is always > 0 so the graph of $y = \ln x$ is always increasing. Since $-\frac{1}{x^2} < 0$ always, the graph of $y = \ln x$ is always concave down (*as the graph of* $\ln x$ *seems to indicate*).

Notice finally that:

$\lim_{x\to\infty} e^x = \infty,\quad \lim_{x\to-\infty} e^x = 0$

$\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to\infty} \ln x = \infty$.

(The last limit seems a little counterintuitive: the graph of $\ln x$ has a slope $\frac{1}{x}$ approaching 0 as $x \to \infty$, so one might feel $\ln x$ should have a horizontal asymptote. But a little thought shous that $\ln x$ can be made as large as we like by taking x large enough. For example:

$$\ln x > 1000 \text{ if } x = e^{\ln x} > e^{1000}.$$

Of course, e^{1000} is very large: $e^{1000} > 10^{333}$ (why?!?). It takes a long while for the graph of $\ln x$ to rise about height 1000, but it does eventually do so. Similarly, $\ln x > 1000000$ if $x = e^{\ln x} > e^{1000000}$.